DOCUMENT RESUME

ED 035 543 SE 007 718

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TITLE An Introduction to Continued Fractions.
INSTITUTION National Council of Teachers of Mathematics, Inc., Washington, D.C.
PUB DATE 64
NOTE 102p.
AVAILABLE FROM National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington, D.C. 20036

EDRS PRICE EDRS Price MF-$0.50 HC Not Available from EDRS.
DESCRIPTORS *Fractions, History, *Instructional Materials, *Mathematical Concepts, *Mathematics, Number Concepts, Problem Solving, *Secondary School Mathematics

ABSTRACT

Provided is an introduction to the properties of continued fractions for the intellectually curious high school student. Among the topics included are (1) Expansion of Rational Numbers into Simple Continued Fractions, (2) Convergents, (3) Continued Fractions and Linear Diophantine Equations of the Type am + bn = c, (4) Continued Fractions and Congruences, (5) Continued Fractions and Determinants, (6) Practical Applications of Continued Fractions, (7) Continued Fractions and Quadratic Irrational Numbers, (8) Continued Fractions and Pell's Equation, (9) Initially Repeating Continued Fractions and Quadratic Equations, and (10) Initially Repeating Continued Fractions and Reduced Quadratic Irrationals. Also included are proofs that show new relationships between bits of familiar mathematics, exercises that demonstrate the properties under investigations, answers to exercises in the appendix, and historical notes on the men who first worked with continued fractions. (RP)

AN INTRODUCTION TO

$$\cfrac{1}{5+\cfrac{1}{1+\cfrac{1}{2}}}$$

$$\ldots + \; a_2 + \ldots$$

$$\frac{3}{0} = 0 + \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{23+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{5+\cfrac{1}{8}}}}}}}}$$

$$\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{1}{a_4+\cfrac{1}{a_5}}}}$$

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AN INTRODUCTION TO

CONTINUED FRACTIONS

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1201 Sixteenth Street, N.W., Washington, D. C., 20036

Library of Congress Catalog Card Number: 64-18579

Printed in the United States of America

PREFACE

My purpose in writing this booklet is to introduce you to a fascinating topic: the properties of continued fractions. I hope that you will not find difficult this informal presentation of a topic that I find most interesting. It is not meant to be difficult. These ideas are usually presented in books dealing with the theory of numbers, and the discussions and proofs found in these books are usually written for college students of mathematics. However, I feel that the intellectually curious high school student should have an opportunity to study a presentation of continued fractions written especially for him. Through a study of continued fractions you should gain increased insight into those properties of our number systems which are being emphasized today in modern courses in mathematics.

CHARLES G. MOORE

CONTENTS

FOREWORD

Since the very appearance of a continued fraction will probably be new to you, it will not seem obvious what results should be expected from a particular discussion. Thus you have here an opportunity to investigate mathematical situations in which creative thinking is called for and is rewarded. I hope you will discover in your investigation of continued fractions many properties that are surprising and exciting. For this reason I have placed some of the more lengthy proofs at the end of the booklet. You may study them after you have become familiar with the property with which the proof is concerned. You should not feel that the proofs which are included in the text are placed there exclusively for the purpose of proving one particular point. One of my principal purposes in presenting the proofs is to help you see new relationships between bits of mathematics with which you are already familiar. Exercises have been included which have been designed for the purpose of helping you appreciate more fully the properties under investigation. Answers to all of the exercises will be found in Appendix B. Historical notes accompany certain discussions to help give you a knowledge of the men who first worked with continued fractions.

I believe that you will find continued fractions fun to work with. It is toward this end that I have used throughout this booklet the more eye-catching elementary form for writing continued fractions instead of adapting one of the more concise notations usually found in books concerned with number theory.

CHAPTER 1

EXPANSION OF RATIONAL NUMBERS INTO CONTINUED FRACTIONS

SIMPLE CONTINUED FRACTIONS

The continued fraction corresponding to a rational number $\frac{r}{s}$ is an expression of the form:

$$\frac{r}{s} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots \frac{1}{a_n}}}} \qquad \text{Ex.} \quad \frac{128}{37} = 3 + \cfrac{1}{2 + \cfrac{1}{5 + \cfrac{1}{1 + \frac{1}{2}}}}$$

In this expression for $\frac{r}{s}$, all of the a's are positive integers with the exception of a_1 which may be negative. These a's are called the *terms* of the continued fraction. The terms of the continued fraction for $\frac{128}{37}$ are the numbers 3, 2, 5, 1, and 2. The continued fraction for $\frac{128}{37}$ can be obtained as follows:

$$\frac{128}{37} = 3 + \frac{17}{37} = 3 + \cfrac{1}{\cfrac{37}{17}} = 3 + \cfrac{1}{2 + \cfrac{3}{17}} = 3 + \cfrac{1}{2 + \cfrac{1}{\cfrac{17}{3}}} =$$

$$3 + \cfrac{1}{2 + \cfrac{1}{5 + \cfrac{2}{3}}} = 3 + \cfrac{1}{2 + \cfrac{1}{5 + \cfrac{1}{\cfrac{3}{2}}}} = 3 + \cfrac{1}{2 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{2}}}}$$

In this booklet we will be dealing with simple continued fractions only; i.e., those where all of the numerators after a_1 are 1. If the numerator of the given fraction is smaller than the denominator, then a_1 is 0.

Ex. $$\frac{3}{7} = 0 + \frac{3}{7} = 0 + \cfrac{1}{\cfrac{7}{3}} = 0 + \cfrac{1}{2 + \cfrac{1}{3}}$$

Exercise Set 1

Expand the following rational numbers into continued fractions.

1. $\frac{75}{31}$ 2. $\frac{29}{8}$ 3. $\frac{25}{11}$ 4. $\frac{13}{19}$ 5. $\frac{79}{50}$ 6. $\frac{50}{79}$ 7. $\frac{1363}{422}$

To evaluate a given continued fraction we may begin at the end and "work our way back up."

Ex. $$1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4}}} = 1 + \cfrac{1}{2 + \cfrac{1}{\cfrac{13}{4}}} = 1 + \cfrac{1}{2 + \cfrac{4}{13}} =$$

$$1 + \cfrac{1}{\cfrac{30}{13}} = 1 + \frac{13}{30} = \frac{43}{30}$$

Exercise Set 2.

Evaluate the following continued fractions.

1. $1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{3}}}}$ 2. $0 + \cfrac{1}{6 + \cfrac{1}{4 + \cfrac{1}{2}}}$ 3. $1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{5}}}}$

If the given rational number is negative, then it is handled as the following example illustrates:

$$-\frac{13}{9} = -2 + \frac{5}{9} = -2 + \cfrac{1}{\cfrac{9}{5}} = -2 + \cfrac{1}{1 + \frac{4}{5}} = -2 + \cfrac{1}{1 + \cfrac{1}{\cfrac{5}{4}}} =$$

$$-2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4}}}.$$

Note: -2 was selected for a_1 because -2 is the largest integer which is less than $-\frac{13}{9}$.

Exercise Set 3

Expand the following rational numbers into simple continued fractions.

1. $-\frac{15}{7}$ 2. $-\frac{23}{13}$ 3. $-\frac{71}{17}$

Historical Note on Continued Fractions

Continued fractions were first investigated by Pietro Cataldi.[1] He was born in Bologna, Italy, in 1548. Cataldi was a mathematics teacher, and his primary mathematical interest was in perfect numbers. A *perfect number* is one which is the sum of its divisors (not counting the number itself as a divisor). For example, 28 is a perfect number because: the divisors of 28 are 1, 2, 4, 7, and 14; and $1 + 2 + 4 + 7 + 14 = 28$. Perhaps you can find more numbers with this property. In the year 1613, Cataldi found approximations for the square roots of numbers by using continued fractions, but he did not make a detailed investigation of continued fractions.[2]

Leonhard Euler, who was an 18th century Swiss mathematician, first used the expression *fractio continua* as a name for continued fractions. The German word for continued fractions is *kettenbrüche* (chain fractions). This name has only been in use since the beginning of the 19th century.[3]

[1] Eaves, Howard. *An Introduction to the History of Mathematics*, New York: Rinehart and Co., 1953. pp. 225-26.
[2] Fink, Karl. *A Brief History of Mathematics*, London: The Open Court Publishing Co., 1910. p. 131.
[3] *Ibid.*, p. 132.

TERMINATING CONTINUED FRACTIONS

All of the continued fractions that we have obtained by expanding rational numbers have come to an end. Would the continued fraction of *every* rational number terminate? Let us again examine the process of expanding $\frac{128}{37}$. First we divide 128 by 37:

$$\begin{array}{r} 3 \\ 37\overline{)128} \\ \underline{111} \\ 17 \end{array}$$

Note that the remainder, 17, must be smaller than 24; for if it is not, then 3 is too small a number for the quotient. Now there is only a limited number of positive integers less than 37. The next division involved in the expansion is as follows:

$$\begin{array}{r} 2 \\ 17\overline{)37} \\ \underline{34} \\ 3 \end{array}$$

What must be true of the remainder, 3? This remainder, of course, must be less than 17, which is less than 37. The next division in the expansion is

$$\begin{array}{r} 5 \\ 3\overline{)17} \\ \underline{15} \\ 2 \end{array}$$

and the remainder, this time, must be a positive integer less than 3. The remainders form a decreasing sequence of positive integers; i.e., 17, 3, 2, $\cdots$; and so we must eventually get a remainder of zero; and at this point the expansion process is terminated. Continued fractions of this type are called *terminating continued fractions.*

> THEOREM 1. *Every rational number can be expanded into a terminating continued fraction.*

Perhaps you would like to write a proof of Theorem 1. Proof No. 1 in Chapter 12 gives a proof of this theorem.

Also as a result of your practice with the exercises in Set 2 you may conclude that the following is true.

> THEOREM 2. *Every terminating continued fraction can be written as a rational number.*

No proof of Theorem 2 will be given.

Could it be possible for a given rational number to be represented by more than one continued fraction? That this is not possible seems rather obvious when one considers our discussion of the remainders involved in the expansion of the rational number $\frac{128}{37}$. However, the proof that rational numbers cannot be represented by more than one continued fraction should be interesting, in that the proof makes use of concepts that are already familiar to you.

DEFINITION: We shall say that *two simple continued fractions are equal if and only if their corresponding terms are equal.*

THEOREM 3. *Every rational number can be represented by only one simple continued fraction.*

Proof. Suppose that

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + \cfrac{1}{a_n}}} \quad \text{and} \quad a'_1 + \cfrac{1}{a'_2 + \cfrac{1}{a'_3 + \cdots + \cfrac{1}{a'_n}}}$$

are both continued fractions that represent the rational number $\frac{r}{s}$.

Then we have

$$\frac{r}{s} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + \cfrac{1}{a_n}}} = a'_1 + \cfrac{1}{a'_2 + \cfrac{1}{a'_3 + \cdots + \cfrac{1}{a'_n}}}.$$

Now a_1 is the largest integer less than $\frac{r}{s}$, and a'_1 is also the largest integer less than $\frac{r}{s}$, so $a_1 = a'_1$. We now have $\frac{r}{s} - a_1 = \frac{r}{s} - a'_1$. Let us set $\frac{r}{s} - a_1 = \frac{r_1}{s_1}$. $\frac{r_1}{s_1}$ is less than 1 because a_1 is the largest integer less than $\frac{r}{s}$.

$$\frac{r}{s} - a_1 = \frac{r_1}{s_1} = \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + \cfrac{1}{a_n}}} = \cfrac{1}{a'_2 + \cfrac{1}{a'_3 + \cdots + \cfrac{1}{a'_n}}}$$

The reciprocal of a positive number less than 1 is greater than 1; therefore $\frac{s_1}{r_1}$ is greater than 1. Since it is also true that if two non-zero numbers are equal, their reciprocals are equal; we can write:

$$\frac{s_1}{r_1} = a_2 + \cfrac{1}{a_3 + \cdots + \cfrac{1}{a_n}} = a'_2 + \cfrac{1}{a'_3 + \cdots + \cfrac{1}{a'_n}}$$

Again, a_2 is the largest integer less than $\frac{s_1}{r_1}$, and a'_2 is also the greatest integer less than $\frac{s_1}{r_1}$; therefore, $a_2 = a'_2$. The same reasoning can be used to show that $a_3 = a'_3$. Therefore the continued fractions

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + \cfrac{1}{a_n}}} \text{ and } a'_1 + \cfrac{1}{a'_2 + \cfrac{1}{a'_3 + \cdots + \cfrac{1}{a'_n}}}$$

which we assumed to represent the number $\frac{r}{s}$ must be equal by our definition of equality. We conclude that every rational number can be represented in only one way as a simple continued fraction.

CHAPTER 2

CONVERGENTS

DEFINITION OF CONVERGENTS

If any of the terms are dropped from the end of a continued fraction, the rational number which is represented by the part retained is called a *convergent*. For the number

$$\frac{r}{s} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{a_5}}}}$$

we may write:

the first convergent is $C_1 = a_1,$

the second convergent is $C_2 = a_1 + \frac{1}{a_2},$

the third convergent is $C_3 = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3}},$

etc.

For $\frac{128}{37} = 3 + \cfrac{1}{2 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{2}}}},$

we may write:

$$C_1 = 3 \qquad\qquad = 3$$

$$C_2 = 3 + \frac{1}{2} \qquad = \frac{7}{2}$$

$$C_3 = 3 + \cfrac{1}{2 + \cfrac{1}{5}} \qquad = \frac{38}{11}$$

$$C_4 = 3 + \cfrac{1}{2 + \cfrac{1}{5 + \cfrac{1}{1}}} \qquad = \frac{45}{13}$$

$$C_5 = 3 + \cfrac{1}{2 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{2}}}} = \frac{128}{37}.$$

Exercise Set 4

Find all of the convergents for the following continued fractions.

1. $2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2}}}}$ 2. $5 + \cfrac{1}{4 + \cfrac{1}{3 + \cfrac{1}{2}}}$ 3. $3 + \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{6 + \cfrac{1}{7}}}}$

We now seek a formula which will enable us to evaluate more rapidly the convergents of a continued fraction. Let C_n represent the nth convergent. Let r_n and s_n represent respectively the numerator and the denominator of C_n.

$C_1 = a_1$; so $r_1 = a_1$, and $s_1 = 1$.

$C_2 = a_1 + \dfrac{1}{a_2} = \dfrac{a_1a_2 + 1}{a_2}$; so $r_2 = a_1a_2 + 1$, and $s_2 = a_2$.

$$C_3 = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3}} = a_1 + \cfrac{1}{\cfrac{a_2a_3 + 1}{a_3}} = a_1 + \frac{a_3}{a_2a_3 + 1} =$$

$$\frac{a_1a_2a_3 + a_1 + a_3}{a_2a_3 + 1} = \frac{a_3(a_1a_2 + 1) + a_1}{a_3a_2 + 1}.$$

Note: $\quad a_1a_2 + 1 = r_2, \quad a_1 = r_1, \quad a_2 = s_2, \quad 1 = s_1.$

Substituting, we have

$$C_3 = \frac{a_3r_2 + r_1}{a_3s_2 + s_1} \quad \text{giving:} \quad \begin{aligned} r_3 &= a_3r_2 + r_1 \\ s_3 &= a_3s_2 + s_1. \end{aligned}$$

By a similar substitution, we can get

$$C_4 = \frac{a_4r_3 + r_2}{a_4s_3 + s_2} \quad \text{giving:} \quad \begin{aligned} r_4 &= a_4r_3 + r_2 \\ s_4 &= a_4s_3 + s_2. \end{aligned}$$

We seem to have a pattern evolving, and we find after studying the expressions for C_3 and C_4 that the formula for C_n seems to be

$$C_n = \frac{r_n}{s_n} = \frac{a_nr_{n-1} + r_{n-2}}{a_ns_{n-1} + s_{n-2}}.$$

If the formula for C_n is to involve r_{n-1}, r_{n-2}, s_{n-1}, and s_{n-2}, we must decide what values to assign to r_0, r_{-1}, s_0, and s_{-1}. We have seen that the formulas are valid for $n = 3$ and for $n = 4$. Are they true for $n = 2$ and for $n = 1$? The formula states $r_2 = a_2r_1 + r_0$. We know $r_2 = a_1a_2 + 1$, and that $r_1 = a_1$. Therefore our formula is applicable to r_2, if we define r_1 to be 1. The formula states $s_2 = a_2s_1 + s_0$. We know $s_2 = a_2$, and $s_1 = 1$. Therefore the formula will be applicable to s_2, if we define s_0 to be 0. The formula states $r_1 = a_1r_0 + r_{-1}$. We know $r_1 = a_1$, and $r_0 = 1$. Therefore the formula will be valid for r_1, if we define r_{-1} to be 0. Again we look at the formula and note that $s_1 = a_1s_0 + s_{-1}$. Now since we have $s_1 = 1$ and $s_0 = 0$, the formula will further be valid for s_1, if we define s_{-1} to be 1. We now adopt the following definitions:

$$r_{-1} = 0, \qquad s_{-1} = 1, \qquad r_0 = 1, \qquad \text{and } s_0 = 0.$$

Our formulas are true for $n = 1, 2, 3,$ and 4. This suggests a proof by mathematical induction. We must show that when the formulas are true for any integer m they are also true for $m + 1$. For the completion of the proof, see Proof No. 2 in Chapter 12.

PRACTICE IN FINDING THE CONVERGENTS OF A CONTINUED FRACTION, USING THE FORMULAS FOR r_n AND s_n

EXAMPLE. To find the convergents for

$$\frac{384}{157} = 2 + \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{8 + \cfrac{1}{2}}}}$$

first, set up a table as shown in Table I.

TABLE I

n	-1	0	1	2	3	4	5
a_n			2	2	4	8	2
r_n							
s_n							

Next, fill in the table for $r_{-1} = 0$, $r_0 = 1$, $s_{-1} = 1$, and $s_0 = 0$, as defined. You always start by filling the $n = -1$ and $n = 0$ columns for r and s as shown in Tables II and III.

TABLE II

n	-1	0	1	2	3	4	5
a_n							
r_n	0	1					
s_n	1	0					

TABLE III

n	-1	0	1	2	3	4	5
a_n			2	2	4	8	2
r_n	0	1	2	5	22	181	384
s_n	1	0	1	2	9	74	157

$$\frac{r_1}{s_1} = \frac{a_1r_0 + r_{-1}}{a_1s_0 + s_{-1}} = \frac{2\cdot 1 + 0}{2\cdot 0 + 1} = \frac{2}{1}$$

$$\frac{r_2}{s_2} = \frac{a_2r_1 + r_0}{a_2s_1 + s_0} = \frac{2\cdot 2 + 1}{2\cdot 1 + 0} = \frac{5}{2}$$

etc.

With a little practice you will see that a convergent table can be filled in very rapidly. The last convergent must be equal to the rational number the continued fraction represents. This gives you a very good check on your arithmetic.

Exercise Set 5

Make convergent tables for the following fractions.

1. $\frac{25}{11}$ 2. $\frac{17}{12}$ 3. $\frac{37}{10}$ 4. $\frac{43}{19}$ 5. $\frac{19}{43}$ 6. $\frac{151}{24}$ 7. $\frac{133}{703}$ 8. $\frac{119}{84}$

Historical Note on Convergents of Continued Fractions

The first mathematician to investigate methods for calculating the convergents of a continued fraction was Daniel Schwenter. Schwenter did this work in 1625.[4] Schwenter, like Cataldi, was interested in perfect numbers.[5] The formulas that we have just developed were first developed by John Wallis[6]: In 1650, Wallis found that

$$\frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \ldots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \ldots}.$$

Lord Brounker[7] rewrote this expression as the following continued fraction:

$$\frac{\pi}{4} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cdots}}}.$$

Now let us examine a particular convergent table. For example, the convergent table for $\frac{167}{61}$ is as shown in Table IV.

Table IV

n	-1	0	1	2	3	4	5	6
a_n			2	1	2	1	4	3
r_n	0	1	2	3	8	11	52	167
s_n	1	0	1	1	3	4	19	61

Note, for example, the differences in the various crisscross products as they are indicated in Table IV.

Example.

$$\begin{aligned}
1 \cdot 1 - 0 \cdot 0 &= +1\\
2 \cdot 0 - 1 \cdot 1 &= -1\\
3 \cdot 1 - 1 \cdot 2 &= +1\\
8 \cdot 1 - 3 \cdot 3 &= -1\\
11 \cdot 3 - 4 \cdot 8 &= +1\\
52 \cdot 4 - 19 \cdot 11 &= -1\\
167 \cdot 19 - 61 \cdot 52 &= +1
\end{aligned}$$

Now go back and examine your other convergent tables and see if you find a similar relationship there. If you find that the differences in the crisscross products as found above give you alternately +1's

[4] Fink, Karl. *A Brief History of Mathematics*. London: Open Court Publishing Co., 1910. p. 131.
[5] Dickson, Leonard Eugene. *History of the Theory of Numbers*, New York: G. E. Stechert and Co., 1934. p. 11.
[6] Fink, *op. cit.*, p. 132.
[7] Eaves, Howard. *An Introduction to the History of Mathematics*, New York: Reinhart and Co., 1953. p. 92.

and -1's, then, what is the formula which is suggested?

Answer: $r_n s_{n-1} - r_{n-1}s_n = (-1)^n$.

Exercise Set 6

Complete the following convergent table (Table V) and calculate $r_n s_{n-1} - r_{n-1}s_n$ for $n = 1$ through $n = 6$.

TABLE V

n	-1	0	1	2	3	4	5	6
a_n			2	3	1	3	4	1
r_n	0	1						
s_n	1	0						

Suppose the formula turns out to be true for all cases. Would it be of any use to us? Look at the last difference of crisscross products in the example: $167 \cdot 19 - 61 \cdot 52 = -1$. This can be written as $167(-19) + 61(+52) = 1$. Thus, -19 and 52 are integral solutions to $167X + 61Y = 1$. It seems that the formula might be useful, so let us see if we can find a way to prove it. Toward this end we examine a general table of convergents (Table VI).

TABLE VI

n	-1	0	1	2	3	4	
a_n			a_1	a_2	a_3	a_4	. . .
r_n	$r_{-1} = 0$	$r_0 = 1$	r_1	r_2	r_3	r_4	
s_n	$s_{-1} = 1$	$s_0 = 0$	s_1	s_2	s_3	s_4	

Evaluating differences of crisscross products in the same manner as before, we get

$$r_0 s_{-1} - r_{-1}s_0 = 1 \cdot 1 - 0 \cdot 0 = 1.$$

$$r_1 s_0 - r_0 s_1 = r_1 \cdot 0 - 1 \cdot s_1 = -1 \cdot s_1.$$

But, $$s_1 = 1;$$

so we have $$r_1 s_0 - r_0 s_1 = -1.$$

$$r_2 s_1 - r_1 s_2 = ?$$

Previously we found the following relationships:

$$r_2 = a_1 a_2 + 1,\ a_1 = r_1,\ r_0 = 1, \text{ and } r_2 = a_2 r_1 + r_0.$$

Also: $s_2 = a_2$, $s_0 = 0$, $s_1 = 1$, and $s_2 = s_2 \cdot 1 + 0$; so $s_2 = a_2 s_1 + s_0$.

Making substitutions of equivalents, we get the following:

$$
\begin{aligned}
r_2s_1 - r_1s_2 &= s_1(a_2r_1 + r_0) - r_1(a_2s_1 + s_0) \\
&= s_1a_2r_1 + s_1r_0 - a_2r_1s_1 - r_1s_0 \\
&= 1\cdot 1 - r_1\cdot 0 \\
r_2s_1 - r_1s_2 &= +1.
\end{aligned}
$$

To evaluate $r_3s_2 - r_2s_3$, we note from previous work that

$$r_3 = a_3r_2 + r_1, \quad \text{and} \quad s_3 = a_3s_2 + s_1.$$

Substituting these values, we have the following:

$$
\begin{aligned}
r_3s_2 - r_2s_3 &= s_2(a_3r_2 + r_1) - r_2(a_3s_2 + s_1) \\
&= s_2a_3r_2 + s_2r_1 - r_2a_3s_2 - r_2s_1 \\
&= s_2r_1 - r_2s_1 \\
&= -1(r_2s_1 - r_1s_2).
\end{aligned}
$$

We have just seen that

$$r_2s_1 - r_1s_2 = 1;$$

therefore, $$r_3s_2 - r_2s_3 = -1.$$

Summarizing our work we have these results:

$$
\begin{aligned}
r_0s_{-1} - r_{-1}s_0 &= +1 \\
r_1s_0 - r_0s_1 &= -1 \\
r_2s_1 - r_1s_2 &= +1 \\
r_3s_2 - r_2s_3 &= -1.
\end{aligned}
$$

The general formula seems to be

$$r_ns_{n-1} - r_{n-1}s_n = (-1)^n.$$

This formula has worked in four cases. Will it work for all cases? The proof, given in Chapter 12 as Proof No. 3, is carried out in considerable detail. It can be used to illustrate how we might go about seeking relationships which will enable us to complete a proof by mathematical induction.

Examine further the convergents you have in your convergent tables. Each convergent, $\frac{r_n}{s_n}$, is a fraction. Can you reduce any of these fractions to lower terms? Try it. You will find that each of these convergents is a fraction in lowest terms. Is this surprising? When we say a fraction is in lowest terms we mean that *there is no integer which will divide evenly both the numerator and the denominator*. Another way of saying this is to state that the numerator and denominator are *relatively prime*.

Let us now prove that *every convergent, $\frac{r_n}{s_n}$, of a continued fraction is always in lowest terms.*

Proof. We have already proved the general formula:

$$r_n s_{n-1} - r_{n-1} s_n = (-1)^n.$$

We now wish to show that r_n and s_n are relatively prime. The statement that r_n and s_n are relatively prime means that there is no integer which will divide evenly both r_n and s_n. Now let us assume that there *is* some integer b ($b \neq 1$) that *will* divide evenly both r_n and s_n. If there exists such an integer, we could write $\frac{r_n}{b} = k_1$ and $\frac{s_n}{b} = k_2$, where k_1 and k_2 are integers. Therefore

$$r_n = bk_1 \quad \text{and} \quad s_n = bk_2$$

and if we substitute these expressions in

$$r_n s_{n-1} - r_{n-1} s_n = (-1)^n,$$

we have
$$(bk_1)s_{n-1} - r_{n-1}(bk_2) = (-1)^n$$

and
$$b(k_1 s_{n-1} - r_{n-1}k_2) = (-1)^n.$$

Note: If n is even, then $(-1)^n$ is $+1$; if n is odd, then $(-1)^n$ is -1. The last equation states that there is an integer $b \neq 1$ which is a divisor of $+1$ or -1. But there is no such integer. Since our supposition has lead us to a false conclusion, we must conclude that our assumption that r_n and s_n have a common divisor is false. Thus we have proved the following theorem:

THEOREM 4. *Each convergent is in lowest terms.*

Now look carefully at the convergents in your tables and see if you can discover any interesting properties that have not already been mentioned. Consider the convergents for $\frac{128}{37}$ which are as given in Table VII.

TABLE VII

C_1	C_2	C_3	C_4	C_5
$\frac{3}{1}$	$\frac{7}{2}$	$\frac{38}{11}$	$\frac{45}{13}$	$\frac{128}{37}$
3	$3\frac{1}{2}$	$3\frac{5}{11}$	$3\frac{6}{13}$	$3\frac{17}{37}$
3	3.5	3.455	3.462	3.459

The convergents for $\frac{225}{157}$ are as given in Table VIII.

TABLE VIII

C_1	C_2	C_3	C_4	C_5
$\frac{1}{1}$	$\frac{3}{2}$	$\frac{10}{7}$	$\frac{43}{30}$	$\frac{225}{157}$
1	$1\frac{1}{2}$	$1\frac{3}{7}$	$1\frac{13}{30}$	$1\frac{68}{157}$
1	1.5	1.4285	1.4333	1.4331

In these examples (Tables VII and VIII) the convergents are alternately greater and less than the rational number the continued fraction represents. The last convergent is, of course, exactly this rational number. We shall now show that this is always the case. We start by considering the difference between any convergent, C_n, and the previous convergent, C_{n-1}.

$$C_n - C_{n-1} = \frac{r_n}{s_n} - \frac{r_{n-1}}{s_{n-1}} = \frac{r_n s_{n-1} - s_n r_{n-1}}{s_{n-1}s_n} = \frac{(-1)^n}{s_{n-1}s_n}$$

$$C_{n+1} - C_n = \frac{r_{n+1}}{s_{n+1}} - \frac{r_n}{s_n} = \frac{r_{n+1}s_n - r_n s_{n+1}}{s_{n+1}s_n} = \frac{(-1)^{n+1}}{s_{n+1}s_n}$$

Now the s's are positive integers, so both $s_{n-1}s_n$ and $s_n s_{n+1}$ will always be positive. If n is even: $(-1)^n$ is $+1$, and $(-1)^{n+1}$ is -1. If n is odd: $(-1)^n$ is -1, and $(-1)^{n+1}$ is -1. So in either case $C_n - C_{n-1}$ and $C_{n+1} - C_n$ will have different signs. Again, the s's are always positive integers. Also each s is larger than the preceding s, so

$$\frac{1}{s_{n+1}s_n} < \frac{1}{s_n s_{n-1}}.$$

And now, making use of the "absolute value" symbols, we have the following:

$$\left|\frac{(-1)^n}{s_{n+1}s_n}\right| < \left|\frac{(-1)^{n-1}}{s_n s_{n-1}}\right|.$$

This means that each convergent is nearer to the value the continued fraction represents, $\frac{r}{s}$ than the preceding convergent. Also, a_1 is always less than $\frac{r}{s}$ $\left(a_1 \text{ being the largest integer less than } \frac{r}{s}\right)$. Also, since the last convergent is equal to $\frac{r}{s}$, we now have proved the following:

> THEOREM 5. *The odd convergents form an increasing sequence of numbers which are all less than* $\frac{r}{s}$ (except that the last number is equal to $\frac{r}{s}$ if n in the number of convergents is odd), *and the even-numbered convergents form a decreasing sequence of numbers which are all greater than the value* $\frac{r}{s}$ (except that the last number is equal to $\frac{r}{s}$ if the number of convergents is even).

We can represent the situation as follows:

$$\frac{r_1}{s_1} < \frac{r_3}{s_3} < \frac{r_5}{s_5} < \frac{r_7}{s_7} < \cdots \quad \frac{r}{s} \quad \cdots < \frac{r_6}{s_6} < \frac{r_4}{s_4} < \frac{r_2}{s_2}.$$

We have been discussing the fact that all of the convergents except the last are different from $\frac{r}{s}$. The question which arises naturally at this point is: How much do the various convergents differ from $\frac{r}{s}$? We now attempt to answer this question. Let us state the question more precisely as: How great is the difference between a given rational number, $\frac{r}{s}$, and the ith convergent, $\frac{r_i}{s_i}$, of its continued fraction? As a start we consider the relationship of the ith term a_i to the rest of the continued fraction.

$$\frac{r}{s} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + a_i + \cfrac{1}{\underbrace{a_{i+1} + \cfrac{1}{a_{i+2} + \cdots + \cfrac{1}{a_n}}}_{R_{i+1}}}}}$$

We see that

$$a_{i+1} + \cfrac{1}{a_{i+2} + \cdots + \cfrac{1}{a_n}}$$

is a terminating continued fraction and thus represents a rational number. Let us call this number R_{i+1}. We can now write the original continued fraction as follows:

$$\frac{r}{s} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + \cfrac{1}{a_i + \cfrac{1}{R_{i+1}}}}}.$$

Considering this continued fraction as having $i + 1$ terms, we note that $\frac{r}{s} = \frac{r_{i+1}}{s_{i+1}}$. Then applying the formula for the $i + 1$th convergent we proceed. We are interested in the size of the difference between $\frac{r}{s}$ and $\frac{r_i}{s_i}$, that is $\frac{r}{s} - \frac{r_i}{s_i}$, so we write:

$$\begin{aligned}\frac{r}{s} - \frac{r_i}{s_i} = \frac{r_{i+1}}{s_{i+1}} - \frac{r_i}{s_i} &= \frac{R_{i+1}r_i + r_{i-1}}{R_{i+1}s_i + s_{i-1}} - \frac{r_i}{s_i} \\ &= \frac{R_{i+1}r_is_i + r_{i-1}s_i - R_{i+1}r_is_i - r_is_{i-1}}{s_i(R_{i+1}s_i + s_{i-1})} \\ &= \frac{r_{i-1}s_i - r_is_{i-1}}{s_i(R_{i+1}s_i + s_{i-1})}.\end{aligned}$$

But $\qquad r_{i-1}s_i - r_is_{i-1} = \pm 1;$

therefore, $\qquad \dfrac{r}{s} - \dfrac{r_i}{s_i} = \dfrac{\pm 1}{s_i(R_{i+1}s_i + s_{i-1})}.$

Now $R_{i+1} > a_{i+1}$ because a_{i+1} is a positive integer, and R_{i+1} is the same positive integer plus the rest of the continued fraction. Note now that, since decreasing the size of the denominator results in making a fraction larger, we have:

$$\frac{r}{s} - \frac{r_i}{s_i} < \left|\frac{1}{s_i(a_{i+1}s_i + s_{i-1})}\right|.$$

But $\qquad a_{i+1}\, s_i + s_{i-1} = s_{i+1};$

so we now have $\qquad \dfrac{r}{s} - \dfrac{r_i}{s_i} < \left|\dfrac{1}{s_is_{i+1}}\right|.$

This inequality states that $\frac{r}{s} - \frac{r_i}{s_i}$ always lies between $-\frac{1}{s_i s_{i+1}}$ and $+\frac{1}{s_i s_{i+1}}$.

To illustrate the use of this formula let us investigate the size of the difference between $\frac{217}{96}$ and its third convergent. The convergent table for $\frac{217}{96}$ is given in Table IX.

TABLE IX

n	-1	0	1	2	3	4	5
a_n			2	3	1	5	4
r_n	0	1	2	7	9	52	217
s_n	1	0	1	3	4	23	96

$$\frac{217}{96} - \frac{r_3}{s_3} < \left|\frac{1}{s_3 s_4}\right|$$

$$\frac{217}{96} - \frac{9}{4} < \left|\frac{\pm 1}{4 \cdot 23}\right|$$

$$\frac{217}{96} - \frac{9}{4} < \left|\frac{\pm 1}{92}\right|$$

Answer: $\frac{217}{96} - \frac{9}{4} < \left|\pm 0.011\right|$.

When giving this type of answer in decimal form, be certain to round-off the decimal upward and not downward. Can you explain why you should not round-off downward?

Exercise Set 7

Investigate the size of the difference between the given number and its specified convergent. Use the formula and state your answer in decimal form.

1. $\frac{1081}{246}$ and its third convergent.
2. $\frac{984}{181}$ and its fourth convergent.

CHAPTER 3

CONTINUED FRACTIONS AND LINEAR DIOPHANTINE EQUATIONS OF THE TYPE $am + bn = c$

DEFINITIONS

An equation of the type $am + bn = c$, where a, b, and c are integers and for which integral solutions are required, is called a *linear diophantine equation* or an *indeterminate equation.* Integers which when substituted for m and n make the equation a true statement are called *solutions for the equation.*

To find solutions, form the fraction $\frac{a}{b}$ or $\frac{b}{a}$. Place the larger value in the numerator. Assume we use $\frac{a}{b}$. Expand this fraction into a continued fraction. Then if there are n terms in the continued fraction, use the formula:

$$r_n s_{n-1} - s_n r_{n-1} = (-1)^n.$$

But $$\frac{a}{b} = \frac{r_n}{s_n}.$$

Substitution gives $as_{n-1} - br_{n-1} = (-1)^n$.

If n is even, then

$$a(s_{n-1}) + b(-r_{n-1}) = 1 \text{ and } s_{n-1} \text{ and } r_{n-1}$$

are solutions for this equation. If n is odd, multiply both sides of the equation by -1. But we want solutions to

$$am + bn = c, \text{ not to } am + bn = 1.$$

To get these solutions, multiply both sides of $a(s_{n-1}) + b(-r_{n-1}) = 1$ by c, getting

$$a(cs_{n-1}) + b(-cr_{n-1}) = c.$$

Thus, solutions to the equation $am + bn = c$ are

$$m = cs_{n-1} \quad \text{and} \quad n = -cr_{n-1}.$$

EXAMPLE. Find integral solutions for the following equation:

$$83m + 118n = 3.$$

$$\frac{118}{83} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \frac{1}{4}}}}}$$

TABLE X

n	-1	0	1	2	3	4	5	6
a_n			1	2	2	1	2	4
r_n	0	1	1	3	7	10	27	118
s_n	1	0	1	2	5	7	19	83

Using $r_n s_{n-1} - r_{n-1} s_n = (-1)^n$ with $n = 6$,

we have the following:

$$r_6 s_5 - r_5 s_6 = (-1)^6$$
$$118(19) - 27(83) = (-1)^6$$
$$83(-27) + 118(19) = 1.$$

Now multiplying both sides of this equation by 3, we have the following:

$$3 \cdot 83(-27) + 3 \cdot 118(19) = 3 \cdot 1$$
$$83(-81) + 118(57) = 3.$$

We see that $m = -81$ and $n = 57$ are solutions to the equation $83m + 118n = 3$. You should check these answers by substituting them into the equation.

Can solutions to equations of this type always be found? Let us investigate this question by considering five integers a, b, c, m, and r with the following properties:

1. a and b are both divisible by some integer $k \neq 1$. This means $a = ku$, and that $b = kv$ for integers u and v.
2. k is not a divisor of c.
3. $am + bn = c$.

Substituting from 1 in the equation $am + bn = c$, we get:

$$(ku)m + (kv)n = c$$

$$k(um + vn) = c.$$

This implies that k is a divisor of c, and this contradicts the second property. This means that integers with the three properties listed above cannot be found. It also means that not all equations of the type $am + bn = c$, where a, b, and c are integers, have integral solutions; $2m + 4n = 3$ is an example of such an equation.

Would you care to try to find integral solutions for the equation $2m + 4n = 3$? If you can find integers which, when substituted for m and n, make the equation a true statement, then 3 is divisible by 2

Exercise Set 3

Using continued fractions, find integral solutions for the following equations.

1. $31x + 11y = 2$ 2. $13x + 54y = 2$ 3. $85x - 30y = 5$
4. $217m - 105n = 6$ 5. $33m + 19n = 100$ 6. $74m - 253n = 1$

Suppose we find a pair of integers that satisfy the equation $am + bn = c$. Are these the only solutions? To answer this question, let us investigate the equation $83m + 118n = 3$ more closely:

$$83m + 118n = 3 \quad \text{and} \quad 83(-81) + 118(57) = 3.$$

Since both of the left-hand members are equal to the same number, we have the following:

$$83m + 118n = 83(-81) + 118(57)$$
$$83m - 83(-81) = 118(57) - 118n$$
$$83(m + 81) = 118(57 - n).$$

Now $83(m + 81)$ and $118(57 - n)$ are equal; therefore, they must have the same factors. But note that 83 and 118 cannot have a common factor because $\frac{118}{83}$ is a convergent, and we proved that all convergents are in lowest terms. So 83 must be a factor of $57 - n$, and 118 must be a factor of $m + 81$. We now have the following equations:

$$m + 81 = 118t \quad \text{and} \quad 57 - n = 83t \text{ for some integer } t.$$
$$m = -81 + 118t \qquad n = 57 - 83t.$$

If m and n are solutions to $83m + 118n = 3$, another pair of integers satisfying the equation can be found by substituting any integer for t in the expressions for m and n. For example, let us by letting t equal 2, find another pair of integers which satisfy the equation $83m + 118n = 3$:

$$m = -81 + 118(2) \quad \text{and} \quad n = 57 - 83(2)$$
$$m = -81 + 236 \qquad n = 57 - 166$$
$$m = 155 \qquad n = -109$$

so $m = 155$ and $n = -109$ are solutions for $83m + 118n = 3$. You should check these values by substituting them into the equation.

Exercise Set 9

Using the integers indicated below as values for t, find a second pair of integers which will satisfy each of the equations in Exercise Set 8 (omit the fourth equation in the set). Answers resulting from the following values for t are given in Appendix A, but any other integer would give valid solutions.

1. $t = 2$ 2. $t = 3$ 3. $t = 1$ 5. $t = -2$ 6. $t = -1$

CHAPTER 4

CONTINUED FRACTIONS AND CONGRUENCES

SOME DEFINITIONS AND EXAMPLES

The expression $a \equiv b(\text{mod } m)$ is read "a is congruent to b modulo m" and means: *a and b have the same remainder when they are divided by m.* The number m is called the *modulus.* For example: $5 \equiv 17(\text{mod } 3)$ is a true statement, because both 5 and 17 have a remainder of 2 when they are divided by 3; but $21 \equiv 33(\text{mod } 10)$ is not a true statement, because 21 and 33 have different remainders upon division by 10.

We can also have congruences involving unknowns such as $ax \equiv b(\text{mod } m)$. A solution for this congruence is a number which when substituted for x will make the congruence a true statement. The number 27 is a solution to the congruence $7x \equiv 9(\text{mod } 5)$, because $7 \cdot 27$, or 189, and 9 both have a remainder of 4 when they are divided by 5. It is also true that if any integral multiple of the modulus is added to a given solution we obtain another solution. In the case just given, 27 was a solution; so $27 + 2 \cdot 5$, or 37, is also a solution. Check: $7 \cdot 37 = 259$; and division of 259 by 5 will also give a remainder of 4.

To find solutions for $ax \equiv b(\text{mod } m)$ by continued fractions, let us consider the continued fraction for $\frac{a}{m}$. The last convergent will of course be $\frac{a}{m}$. If there are n convergents, let us substitute $r_n = a$ and $s_n = m$ in the formula $r_n s_{n-1} - r_{n-1} s_n = (-1)^n$, getting the following:

$$as_{n-1} - r_{n-1}m = (-1)^n$$

$$s_{n-1} = \frac{r_{n-1}m + (-1)^n}{a}$$

$$ax = as_{n-1} = a\left(\frac{r_{n-1}m + (-1)^n}{a}\right) = r_{n-1}m + (-1)^n.$$

Now divide both $r_{n-1}m + (-1)^n$ and 1 by the modulus m.

$$\begin{array}{r|l} & r_{n-1} \\ \hline m) & r_{n-1}m + (-1)^n \\ & r_{n-1}m \\ \hline & (-1)^n \end{array} \qquad \begin{array}{r|l} & 0 \\ \hline m) & 1 \\ & 0 \\ \hline & 1 \end{array}$$

The remainders are $(-1)^n$ and 1. If n is even, $(-1)^n$ is 1; and s_{n-1} is a solution for $ax \equiv 1(\text{mod } m)$. If n is odd, consider $-s_{n-1}$. Substituting $-s_{n-1}$ for x, we have the following:

$$ax = a(-s_{n-1}) = a\left(-\frac{r_{n-1}m + (-1)^n}{a}\right) = -r_{n-1}m + (-1)^{n+1}.$$

Divide $-r_{n-1}m + (-1)^{n+1}$ by the modulus m.

$$\begin{array}{r|l} & -r_{n-1} \\ \hline m) & -r_{n-1}m + (-1)^{n+1} \\ & -r_{n-1}m \\ \hline & (-1)^{n+1} \end{array}$$

If n is odd, then $(-1)^{n+1}$ is 1.

We conclude that if the number of convergents is even, s_{n-1} is a solution for $ax \equiv 1(\text{mod } m)$; and if n is odd, $-s_{n-1}$ is a solution for $ax \equiv 1(\text{mod } m)$. But we need a solution for $ax \equiv b(\text{mod } m)$. So, if $a(s_{n-1}) \equiv 1(\text{mod } m)$ is a true statement, let us multiply both sides of this congruence by b, getting $a(bs_{n-1}) \equiv b(\text{mod } m)$. Then bs_{n-1} is a solution, if n is even; and $-bs_{n-1}$ is a solution to $ax \equiv b(\text{mod } m)$, if n (the number of convergents for $\frac{a}{m}$) is odd.

Let us now use continued fractions to solve a convergent.

Example. Find a solution for $11x \equiv 13(\text{mod } 7)$.

TABLE XI

$$\frac{11}{7} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{3}}}$$

n	-1	0	1	2	3	4
a_n			1	1	1	3
r_n	0	1	1	2	3	11
s_n	1	0	1	1	2	7

In this case n is 4; and since n is even, s_{n-1} (or s_3 which is 2) is a solution to

$$11x \equiv 1(\text{mod } 7);$$

so we write
$$11 \cdot 2 \equiv 1(\text{mod } 7).$$

Now multiplying by 13, we have

$$11(2 \cdot 13) \equiv 13(\text{mod } 7).$$

So $2 \cdot 13$, or 26, is a solution for $11x \equiv 13(\text{mod } 7)$. You may check by dividing both $26 \cdot 11$ and 13 by 7. You will see that the remainder is 6 in both cases. The general solution is $26 + k \cdot 7$ with k being any integer. Let $k = 2$, and find another solution and check it.

If the solution is negative as a result of the number of terms in the continued fraction being odd, then add to this a multiple of the modulus large enough to give a positive solution. The positive solutions are easier to check, but you should also investigate the problem of checking your negative solutions.

Exercise Set 10

Find a solution for the following convergences, using continued fractions and also show the general solution. Use the general solution to find a second solution and check this answer.

1. $7x \equiv 9(\text{mod } 5)$ **2.** $17x \equiv 19(\text{mod } 12)$

3. $13x \equiv 21(\text{mod } 9)$ **4.** $29x \equiv 48(\text{mod } 11)$

If you would like to know more about congruences, you can find a very good discussion of this topic in the book by Carl H. Denbow and Victor Goedicke: *Foundations of Mathematics*. New York: Harper and Brothers, 1959. Chapter 15.

CHAPTER 5

CONTINUED FRACTIONS AND DETERMINANTS

AN INTERESTING QUESTION

Would it be possible to find the nth convergent for a continued fraction without finding first all of the preceding convergents? Mathematicians worked with continued fractions for many years looking for a way to do this. It can be done, and in doing it you will discover an interesting relationship between continued fractions and determinants. If you have not studied determinants, your teacher will be glad to help you with the elementary operations that are referred to here.

Let us first consider the problem of finding the numerator, r_n, of the nth convergent. We shall start our investigation by writing the equation $r_n = a_n r_{n-1} + r_{n-2}$ (which is the same as $a_n r_{n-1} + r_{n-2} = r_n$) for the numerator of the first five convergents of a continued fraction:

$$\begin{aligned} a_1 r_0 + r_{-1} &= r_1 \\ a_2 r_1 + r_0 &= r_2 \\ a_3 r_2 + r_1 &= r_3 \\ a_4 r_3 + r_2 &= r_4 \\ a_5 r_4 + r_3 &= r_5. \end{aligned}$$

Now rearrange these equations in the following manner:

$$\begin{aligned}
r_{-1} + a_1 r_0 - r_1 &= 0\\
r_0 + a_1 r_1 - r_2 &= 0\\
r_1 + a_3 r_2 - r_3 &= 0\\
r_2 + a_4 r_3 - r_4 &= 0\\
r_3 + a_5 r_4 - r_5 &= 0.
\end{aligned}$$

We know by definition that $r_{-1} = 0$, $r_0 = 1$. Therefore $r_1 = a_1$ and, using $-r_1 = -a_1$ for our first equation, we now have the following equations:

$$\begin{aligned}
-r_1 &= -a_1\\
a_2 r_1 - r_2 &= -1\\
r_1 + a_3 r_2 - r_3 &= 0\\
r_2 + a_4 r_3 - r_4 &= 0\\
r_3 + a_5 r_4 - r_5 &= 0.
\end{aligned}$$

Here we have five linear equations in the five unknowns, r_1 through r_5.

We can solve for any one of the unknowns by using determinants. In particular, let us solve for r_5:

$$r_5 = \frac{\begin{vmatrix} -1 & 0 & 0 & 0 & -a_1 \\ a_2 & -1 & 0 & 0 & -1 \\ 1 & a_3 & -1 & 0 & 0 \\ 0 & 1 & a_4 & -1 & 0 \\ 0 & 0 & 1 & a_5 & 0 \end{vmatrix}}{\begin{vmatrix} -1 & 0 & 0 & 0 & 0 \\ a_2 & -1 & 0 & 0 & 0 \\ 1 & a_3 & -1 & 0 & 0 \\ 0 & 1 & a_4 & -1 & 0 \\ 0 & 0 & 1 & a_5 & -1 \end{vmatrix}}.$$

If we think in terms of evaluating the denominator determinant by minors, it becomes apparent that the value of this determinant is $(-1)^5$ in this case or $(-1)^n$ in the general case. Now let us place the first column of the numerator determinant in the first position by interchanging successively columns 5 and 4, 4 and 3, 3 and 2, 2 and 1. Recall that interchanging two columns of a determinant results in the sign of the determinant being changed. So to get the last column in the first position, we require four or $n - 1$ changes of the sign. Now let us change the sign of the elements in the new first column. It is also true that changing the signs of the elements in a column changes the sign of the determinant. We have made $n - 1 + 1$ changes of the sign of the determinant, which is the same as making the alterations of the determinant mentioned above and then multiplying the determinant by $(-1)^n$. But remember that the denominator determinant is equal to $(-1)^n$ also, so these values cancel out regardless of whether n is even or odd. So for r_5 (the numerator of the fifth convergent) we have the following:

$$r_5 = \begin{vmatrix} a_1 & -1 & 0 & 0 & 0 \\ 1 & a_2 & -1 & 0 & 0 \\ 0 & 1 & a_3 & -1 & 0 \\ 0 & 0 & 1 & a_4 & -1 \\ 0 & 0 & 0 & 1 & a_5 \end{vmatrix}.$$

The form of the determinant is easy to remember, and it is not difficult to evaluate by minors with respect to the first column. Determinants of this type are called *continuants* or *cumulants*.[3]

To find the determinant for s_5 (the denominator of the fifth convergent), we proceed in the same way as we did for r_5. Doing so, we get the following:

$$s_5 = \begin{vmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & a_2 & -1 & 0 & 0 \\ 0 & 1 & a_3 & -1 & 0 \\ 0 & 0 & 1 & a_4 & -1 \\ 0 & 0 & 0 & 1 & a_5 \end{vmatrix}.$$

But this determinant can be simplified by expanding by minors with respect to the first column. Doing so, we get the following:

$$s_5 = \begin{vmatrix} a_2 & -1 & 0 & 0 \\ 1 & a_3 & -1 & 0 \\ 0 & 1 & a_4 & -1 \\ 0 & 0 & 1 & a_5 \end{vmatrix}.$$

This process of finding a convergent without first finding the previous convergents will now be illustrated with an example. Using deter-

[3]Perron, Oskar. *Die Lehre von den Kettenbrüchen.* New York: Chelsea Publishing Co., 1950. p. 11.

minants we evaluate the fourth convergent of the continued fraction for $\frac{205}{74}$. All we need are the first four terms: a_1, a_2, a_3, and a_4.

EXAMPLE.

$$\frac{205}{74} = 2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{2 + \frac{5}{6}}}}$$

$$C_4 = \frac{r_4}{s_4} = \frac{\begin{vmatrix} 2 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ 1 & 3 & -1 \\ 0 & 1 & 2 \end{vmatrix}} = \frac{2\begin{vmatrix} 1 & -1 & 0 \\ 1 & 3 & -1 \\ 0 & 1 & 2 \end{vmatrix} - 1\begin{vmatrix} -1 & 0 & 0 \\ 1 & 3 & -1 \\ 0 & 1 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ 1 & 3 & -1 \\ 0 & 1 & 2 \end{vmatrix}} =$$

$$\frac{2\,(6+1+2) - 1\,(-6-1)}{6+1+2} = \frac{2 \cdot 9 + 7}{9} = \frac{25}{9}$$

We can show that this really is the fourth convergent by completing a convergent table (Table XII) for the first four convergents of the continued fraction for $\frac{205}{74}$. We already have the first four a's.

TABLE XII

n	-1	0	1	2	3	4
a_n			2	1	3	2
r_n	0	1	2	3	11	25
s_n	1	0	1	1	4	9

Exercise Set 11

Using determinants, find the indicated convergents of the continued fractions for the following numbers. Check your answers by making a table of convergents.

1. Third convergent for $\frac{34}{15} = ?$
2. Third convergent for $\frac{86}{27} = ?$
3. Fourth convergent for $\frac{31}{12} = ?$

Of course, the fact that it is easier to get the desired convergents by first constructing a convergent table is not important. Our objective is to get new ideas and observe new relationships.

CHAPTER 6

SOME PRACTICAL APPLICATIONS OF CONTINUED FRACTIONS

PART 1: A METHOD FOR FINDING THE TERMS

In many of the practical applications of continued fractions it is necessary to write the continued fraction for rational numbers in which the numerator and denominator are quite large. To do this you need a convenient method for finding the terms. Observe the way the divisions involved in the expansion of $\frac{584}{169}$ are arranged:

EXAMPLE.
$$\frac{584}{169} = 3 + \cfrac{1}{2 + \cfrac{1}{5 + \cfrac{1}{7 + \cfrac{1}{2}}}}$$

$$\begin{array}{rrrrrr}
 & 3 & & & & \\
169 & \overline{)584} & & & & \\
 & 507 & 2 & & & \\
 & \overline{77} & \overline{)169} & & & \\
 & & 154 & 5 & & \\
 & & \overline{15} & \overline{)77} & & \\
 & & & 75 & 7 & \\
 & & & \overline{2} & \overline{)15} & \\
 & & & & 14 & 2 \\
 & & & & \overline{1} & \overline{)2} \\
 & & & & & 2 \\
 & & & & & \overline{0}
\end{array}$$

If you are interested primarily in obtaining the terms of the continued fraction, all you need to do is carry out the divisions as shown at the lower-left in the example for $\frac{584}{169}$. You then keep dividing each remainder into the previous divisor until you get a remainder of zero. The quotients you have obtained are then the terms of the continued fraction, as indicated by the arrows, and they can be placed directly in a convergent table to be used in calculating the convergents.

PART 2: USING CONTINUED FRACTIONS TO SOLVE GEAR-RATIO PROBLEMS

Continued fractions become very practical mathematical tools for a machinist who works with lathes or other instruments where shafts are made to turn by means of gear wheels. The reason for using continued fractions in such situations is that most gear wheels used in machine shops have no less than 20 teeth and no more than 100 teeth. A gear wheel with less than 20 teeth does not mesh smoothly, and if there are more than 100 teeth the teeth are so small that they are impractical.

If a machinist wants two shafts, A and B, to be connected by two gear wheels so that shaft A revolves 37 times every time shaft B revolves 51 times, he places a gear wheel with 37 teeth on shaft B and a gear wheel with 51 teeth on shaft A. Then if the gear wheels mesh, the ratio of the number of revolutions of A to the number of revolutions of B after any period of time will be $\frac{37}{51}$. Remember that the shaft driven by the gear with the larger number of teeth turns more slowly than the shaft driven by the gear wheel with the smaller number of teeth.

First Example

The problem is that in certain cases the machinist is asked to set up his machine so that the ratio of the number of revolutions of one shaft to the number of revolutions of the other after any period of time is, for example, 0.6713. Now this desired ratio, which was given in decimal form, can be expressed as a fraction: $\frac{6713}{10{,}000}$. Two gear wheels, one with 6713 teeth and the other with 10,000 teeth, would do the job; however, we must remember that the number of teeth must be no more than 100 and no less than twenty. So the machinist's problem is to *find a fraction which is very near to* $\frac{6713}{10{,}000}$, *but whose members are no more than* 100 *and no less than* 20. This is done by expanding the fraction, $\frac{6713}{10{,}000}$, into a continued fraction and forming a table of convergents. Recall from a previous discussion that each successive convergent is nearer to the number the continued fraction represents than the preceding one. Therefore the machinist keeps evaluating the convergents until one gives a numerator or denominator greater than 100. He then selects the immediately preceding convergent as the fraction he will use to approximate the desired ratio which was given as a decimal.

Let us now solve the problem described above.

$$\frac{6713}{10{,}000} = 0 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{23 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{5 + \frac{1}{8}}}}}}}}$$

TABLE XIII

n	-1	0	1	2	3	4	5	6	7	8	9
a_n			0	1	2	23	1	1	1	5	8
r_n	0	1	0	1	2	47	$\boxed{49}$	96	145	821	6,713
s_n	1	0	1	1	3	70	$\boxed{73}$	143	216	1,223	10,000

We select the fifth convergent, $\frac{49}{73}$, as our approximation to 0.6713. Using the formula for the size of the difference between the value of a continued fraction and its fifth convergent, we have the following inequalities:

$$\left|0.6713 - \frac{49}{73}\right| < \left|\frac{1}{(73)(143)}\right|,$$

$$\left|0.6713 - \frac{49}{73}\right| < \left|\frac{1}{10{,}439}\right|,$$

$$\left|0.6713 - \frac{49}{73}\right| < \left|0.000095\right|.$$

According to the last inequality, we can see that the error in using two gears with 49 teeth and 73 teeth, respectively, instead of two gears with 6,713 teeth and 10,000 teeth, respectively, is less than 0.000095. To see exactly how large the error is, you should divide 49 by 73 and subtract the quotient from 0.6713.

SECOND EXAMPLE

Find a rational number by using continued fractions which would be a good substitute for a machinist to use in setting up a gear ratio instead of the decimal 0.3847.

$$0.3847 = \frac{3{,}847}{10{,}000}$$

```
                                   a1
              0  <-----
10,000 )3,847                          a2
         0        2  <-----
 3,847 )10,000                             a3
         7,694          1  <-----
         2,306 )3,847                          a4
                2,306        1  <-----
                1,541 )2,306                       a5
                       1,541        2  <-----
                         765 )1,541                    a6
                              1,520   69  <-----
                                 11 )765
                                     60
                                    105                  a7
                                     99   1  <-----
                                      6 )11                 a8
                                          6  1  <-----
                                          5 )6                 a9
                                             5  5  <-----
                                             1 )5
                                                5
                                                0
```

TABLE XIV

n	-1	0	1	2	3	4	5	6	7	8	9
a_n			0	2	1	1	2	69	1	1	5
r_n	0	1	0	1	1	2	5	347	352	699	3,847
s_n	1	0	1	2	3	5	13	902	915	1,817	10,000

We notice that in the fifth convergent, $\frac{5}{13}$, both the numerator and denominator are less than 20, so this is not a suitable rational approximation to 0.3847. The sixth convergent is $\frac{347}{902}$ and is unsuitable because its terms are greater than 100. We now notice that a_5 is 2, and a_6 is 69, an unusually large jump. From a previous discussion we know that $\frac{347}{902}$ is nearer to 0.3847 than is $\frac{5}{13}$. So instead of using 69 as a multiplier, we use the largest integer between 2 and 69 that will result in both the numerator and denominator being no greater than 100. Accordingly we select 7 as our multiplier, and we find that $7 \cdot 5 + 2 = 37$, and $7 \cdot 13 + 5 = 96$; so we choose $\frac{37}{96}$ as our rational approximation to 0.2847. Dividing, we find $\frac{37}{96} = 0.385+$.

You may have anticipated the next question. What happens if we want a suitable rational approximation to a number such as 0.0327?

The method we have been using will yield convergents $C_4 = \frac{2}{61}$ and $C_5 = \frac{5}{153}$. In this case the machinist may have to use compound gears. This means that he will select a convergent such that the numerator is larger than 20, find factors of the denominator which are no less than 20 or no more than 100, and use these in a proper arrangement to achieve the desired ratio.

If you would like to read a good discussion of compound gears see: John M. Christman's *Shop Mathematics*. New York: The Macmillian Company, 1946.

Exercise Set 12

By using continued fractions, find a rational number for a machinist to use in setting up a gear ratio as a suitable substitute for each of the decimals given below.

1. 0.639 **2.** 0.547 **3.** 0.713 **4.** 0.3847

PART 3: FINDING RATIONAL APPROXIMATIONS TO THE NUMBERS π AND e

You may find rational approximations to irrational numbers such as π and e by employing the methods described earlier in Parts 1 and 2. Consider first the following decimal approximations:

$$\pi \approx 3.1\,4\,1\,5\,9\,2\,6\,5\,3\,5$$

$$e \approx 2.7\,1\,8\,2\,8\,1\,8\,2\,9\,4\,5\,9.$$

Of course you can obtain a rational number approximating π by taking, for example, the first five digits of the above decimal; i.e., 3.1 4 1 5, and writing this as a rational number as follows:

$$3.1415 = 3\,\frac{1{,}415}{10{,}000} = \frac{3{,}1415}{10{,}000}.$$

You could also find a rational number which is a better approximation to π by using the first six digits of the given decimal; i.e., 3.1 4 1 5 9. This would give:

$$3.14159 = 3\,\frac{14{,}159}{100{,}000} = \frac{314{,}159}{100{,}000}.$$

However, by using continued fractions you can find rational numbers which approximate the value π better than 3.1 4 1 5 does, and which have numerators less than 314,159 and denominators less than 100,000. As an illustration let us solve now the problem just suggested.

Problem 1

Find five rational numbers each of which is a closer approximation to π than is 3.1415, and each of which has a numerator less than 314,159 and a denominator less than 100,000.

Solution: The decimals 3.1415 and 3.14159 are both approximations to π, and 3.14159 is a closer approximation to π than is 3.1415. The rational numbers corresponding to these decimals are

$$\frac{31{,}415}{10{,}000} \quad \text{and} \quad \frac{314{,}159}{100{,}000}.$$

Let us set up the convergent table for the number $\frac{314{,}159}{100{,}000}$. This convergent table is given in Table XV.

TABLE XV

n	-1	0	1	2	3	4	5	6	7	8
a_n			3	7	15	1	25	1	7	4
r_n	0	1	3	22	333	355	9,208	9,563	76,149	314,159
s_n	1	0	1	7	106	113	2,931	3,044	24,239	100,000

Did you notice that the second convergent is $\frac{22}{7}$ which is, probably, the first rational approximation to π that you learned?

Now 3.14159 is nearer to π than is 3.1415, therefore any number nearer to 3.14159 than is 3.1415 will be nearer to π than is 3.1415. To find the required rational numbers, you only need to write the convergents as decimals until you find one that is nearer than 3.1415 to 3.14159. The difference between 3.1415 and 3.14159 is 0.00009. Rounding off decimals at the fifth decimal place we have the following calculations:

$$C_1 = \frac{3}{1} = 3.00000,$$
$$\text{and} \quad 3.14159 - 3.00000 = 0.14159.$$

$$C_2 = \frac{22}{7} \approx 3.14286,$$
$$\text{and} \quad 3.14286 - 3.14159 = 0.00127.$$

$$C_3 = \frac{333}{106} \approx 3.14151,$$
$$\text{and} \quad 3.14159 - 3.14151 = 0.00008.$$

Note that 0.00008 is less than 0.00009, which shows that $\frac{333}{106}$ is nearer than 3.1415 is to 3.14159. Since each convergent in the convergent table for 3.14159 is nearer to 3.14159 than the preceding convergent, we have the result that C_3, C_4, C_5, C_6, and C_7 are rational numbers with the required properties; i.e., each of the rational numbers $\frac{333}{106}$, $\frac{355}{113}$, $\frac{9208}{2931}$, $\frac{9563}{3044}$, and $\frac{76149}{24239}$ is nearer than 3.1415 is to π and each has a numerator less than 314,159 and a denominator less than 100,000.

It should be noted that this same process can be used to find other rational approximations to any irrational number when a decimal approximation is given.

Excercise 12a.

Find five rational numbers such that each is nearer than 2.7183 to the number e, and such that the numerator of each is less than 271,828 and the denominator of each is less than 100,000.

Answer for Exercise 12a.

The convergent table for $\frac{271\,828}{100\,000}$, or 2.71828, is given in Table XVI.

TABLE XVI

n	-1	0	1	2	3	4	5	6
a_n			2	1	2	1	1	4
r_n	0	1	2	3	8	11	19	87
s_n	1	0	1	1	3	4	7	32

(Table XVI continued below)

n	7	8	9	10	11	12	13
a_n	1	1	6	10	1	1	2
r_n	106	193	1,264	12,833	14,097	26,930	67,957
s_n	39	71	465	4,721	5,186	9,907	25,000

Note: $C_{13} = \dfrac{67{,}957}{25{,}000} = \dfrac{271{,}828}{100{,}000}$.

The difference between 2.71828 and 2.7182 is

$$2.71828 - 2.7182 = 0.00008.$$

$$C_7 = \frac{106}{39} \approx 2.71795,$$

and $\quad 2.71828 - 2.71795 = 0.00033.$

$$C_8 = \frac{193}{71} \approx 2.71831,$$

and $\quad 2.71831 - 2.71828 = 0.00003.$

We note that 0.00003 is less than 0.00008, which means that $\frac{193}{17}$ is nearer than 2.7182 is to 2.71828. Therefore the required rational numbers are

$$\frac{193}{71}, \frac{1{,}264}{465}, \frac{12{,}833}{4721}, \frac{14{,}097}{5186}, \text{ and } \frac{26{,}930}{9907}.$$

PART 4: CONTINUED FRACTIONS AND THE SLIDE RULE

You can see an interesting relationship between continued fractions and settings on a slide rule if you refer to the convergent table for $\frac{314{,}159}{100{,}000} = 3.14159$ which we used as an approximation to π in Part 3. Instead of using long division to check that the rational numbers C_4 through C_7 really are approximations to π, do this division on a slide rule. After dividing the numerator by the denominator of several of these numbers; e.g., $\frac{333}{106}$, $\frac{355}{113}$, $\frac{9{,}208}{2{,}931}$, $\frac{9{,}563}{3{,}044}$, and $\frac{76{,}149}{24{,}239}$, you will be convinced that they are indeed all very good approximations to π, which is usually indicated on the D scale of a slide rule.

For example, to divide 355 by 113 on a slide rule, place the hairline of the indicator over 355 on the D scale. Then place 113 on the C scale under the hairline. The result of the division will be read on the D scale under the 1 at the left of the C scale.

Exercise Set 12b

Perform the following divisions on a slide rule (the numbers below were derived as solutions for Problem 1).

1. $\frac{333}{106}$ 2. $\frac{355}{113}$ 3. $\frac{9{,}208}{2{,}931}$ 4. $\frac{9{,}563}{3{,}044}$ 5. $\frac{76{,}149}{24{,}239}$

Answer for Exercise 12b

The result of each division should read π on the D scale.

CHAPTER 7

CONTINUED FRACTIONS AND QUADRATIC IRRATIONAL NUMBERS

SOME INTERESTING RELATIONSHIPS

We shall now investigate some of the interesting relationships between continued fractions and quadratic irrational numbers. These are numbers of the form $\frac{A + \sqrt{B}}{C}$ where A and C are integers, $C \neq 0$, and B is a positive integer such that $\sqrt{B}$ is irrational.

You should first fix in mind the concept of the *integer part* of a number. This concept will be very useful to you as you read the rest of this booklet. Be certain you understand the statements below which were chosen to help make the idea clear.

> DEFINITION: *The integer part of a number is the largest integer which is less than or equal to the number.*

1. The integer part of $3\frac{1}{2}$ is 3.
2. The integer part of 6.75 is 6.
3. The integer part of $\sqrt{5}$ is 2 because $\sqrt{5}$ is between 2 and 3.
4. The integer part of $\sqrt{17} + 5$ is 9.

5. The integer part of $\sqrt{13} - 1$ is 2.

6. The integer part of $\dfrac{4 + \sqrt{15}}{3}$ is 2.

7. The integer part of $-3\frac{1}{4}$ is -4.

8. The integer part of $\dfrac{2 - \sqrt{7}}{4}$ is -1.

You can find the integer part of a quadratic irrational quickly if you first think of the integer part of the irrational part. The integer part of a number, N, is often written $[N]$; for example, $[2.07] = 2$. However, this notation will not be used in this booklet.

Exercise Set 13

Find the integer part of each of the following numbers:

1. 9.63 2. $\sqrt{53}$ 3. $\dfrac{3 + \sqrt{17}}{2}$ 4. -5.2 5. $\dfrac{9 - \sqrt{10}}{2}$

Before going further, it might be well to review the process of rationalizing the denominator of a fraction of the form $\dfrac{C}{D + \sqrt{E}}$. You may rationalize the denominator by multiplying both numerator and denominator by the conjugate of the denominator. The result of this multiplication will be a fraction whose denominator contains no irrational number.

EXAMPLE 1. Rationalize the denominator in the fraction $\dfrac{2}{3 - \sqrt{2}}$.

$$\frac{2}{3 - \sqrt{2}} = \frac{2\,(3 + \sqrt{2})}{(3 - \sqrt{2})\,(3 + \sqrt{2})} = \frac{6\,(3 + \sqrt{2})}{9 - 4} = \frac{6\,(3 + \sqrt{2})}{5}.$$

A study of the relationships between continued fractions and irrational numbers can be instrumental in helping you gain a deeper insight into the relationships between rational and irrational numbers. In this chapter we will be dealing with quadratic irrationals only. You will first learn how to expand quadratic irrationals into continued fractions, using what we shall call the *three-step process*.

The three-step process will now be illustrated by developing the continued fraction for $\sqrt{8}$. Numbers such as $\sqrt{8}$, which are of the form $\sqrt{B}$ where B is a positive integer, are called *pure quadratic irrationals*.

Step 1. The integer part of $\sqrt{8}$ is 2.

Write: $\sqrt{8} = 2 + \sqrt{8} - 2$
or $\sqrt{8} = 2 + (-2 + \sqrt{8})$.

Let us call this step *splitting* the number.

Step 2. Write $-2 + \sqrt{8}$ as $\dfrac{1}{\dfrac{1}{-2 + \sqrt{8}}}$.

We shall call this step the *flipping* operation.

Step 3. Rationalize the denominator in $\dfrac{1}{-2 + \sqrt{8}}$:

$$\frac{1}{(-2 + \sqrt{8})}\frac{(-2 - \sqrt{8})}{(-2 - \sqrt{8})} = \frac{-2 - \sqrt{8}}{4 - 8} = \frac{-2 - \sqrt{8}}{-4} = \frac{2 + \sqrt{8}}{4}.$$

We now have $\sqrt{8} = 2 + \dfrac{1}{\dfrac{2 + \sqrt{8}}{4}}$.

We keep repeating these three steps. The three steps of the three-step process are as follows:

(1) split (2) flip (3) rationalize.

Now apply the three steps to $\dfrac{2 + \sqrt{8}}{4}$, as shown below.

Step 1. The integer part of $\dfrac{2 + \sqrt{8}}{4}$ is 1.

Split $\dfrac{2 + \sqrt{8}}{4}$, getting: $1 + \left(\dfrac{2 + \sqrt{8}}{4} - 1\right) = 1 + \dfrac{2 + \sqrt{8} - 4}{4}$

$$= 1 + \frac{-2 + \sqrt{8}}{4}.$$

Step 2. Flip $\dfrac{-2 + \sqrt{8}}{4}$, getting: $\dfrac{1}{\dfrac{4}{-2 + \sqrt{8}}}$.

Step 3. Rationalize the denominator in $\dfrac{4}{-2 + \sqrt{8}}$, getting:

$$\frac{4}{(-2 - \sqrt{8})} \cdot \frac{-2 - \sqrt{8}}{-2 - \sqrt{8}} = \frac{4(-2 - \sqrt{8})}{4 - 8} = \frac{4(-2 - \sqrt{8})}{-4} =$$

$$\frac{2 + \sqrt{8}}{1}.$$

Now we can write: $\sqrt{8} = 2 + \cfrac{1}{1 + \cfrac{1}{\frac{2 + \sqrt{8}}{1}}}$.

Now perform the three-step process on $\frac{2 + \sqrt{8}}{1}$ as shown below.

Step 1. Split $\frac{2 + \sqrt{8}}{1}$. The integer part of $\frac{2 + \sqrt{8}}{1}$ is 4.

$$\frac{2 + \sqrt{8}}{1} = 4 + \frac{2 + \sqrt{8}}{1} - 4 = 4 + \frac{2 + \sqrt{8} - 4}{1} = 4 + \frac{-2 + \sqrt{8}}{1}$$

Step 2. Flip $\frac{-2 + \sqrt{8}}{1}$, getting: $\cfrac{1}{\cfrac{1}{-2 + \sqrt{8}}}$.

Step 3. Rationalize the denominator in $\frac{1}{-2 + \sqrt{8}}$, getting:

$$\frac{1}{(-2 + \sqrt{8})} \cdot \frac{(-2 - \sqrt{8})}{(-2 - \sqrt{8})} = \frac{-2 - \sqrt{8}}{4 - 8} = \frac{2 + \sqrt{8}}{4}.$$

Now we have: $\sqrt{8} = 2 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{\frac{2 + \sqrt{8}}{4}}}}$.

We could now repeat the three-step process on $\frac{2 + \sqrt{8}}{4}$, but looking back over our work we note that we have already applied the three steps to $\frac{2 + \sqrt{8}}{4}$, and that the next two terms that arose were 1 and 4. Then if we applied the three steps to $\frac{2 + \sqrt{8}}{4}$ again, we should again get the terms 1 and 4. Therefore the terms are repeating. The continued fraction for $\sqrt{8}$ will never terminate, and we can now express $\sqrt{8}$ as the following continued fraction.

$$\sqrt{8} = 2 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{4} + \cdots}}}}}$$

Let us now examine the convergent table (Table XVII) for this continued fraction.

TABLE XVII

n	-1	0	1	2	3	4	5	6	7	8
a_n			2	1	4	1	4	1	4	1
r_n	0	1	2	3	14	17	82	99	478	577
s_n	1	0	1	1	5	6	29	35	169	204

The fifth convergent, $\frac{82}{29}$, = 2.82758+

The sixth convergent, $\frac{99}{35}$, = 2.82857+

The seventh convergent, $\frac{478}{169}$, = 2.82840+

The eighth convergent, $\frac{577}{204}$, = 2.82843+

From a standard set of tables, we read: $\sqrt{8} = 2.828427+$.

Exercise Set 14

Using the three-step process, expand the following pure quadratic irrationals into continued fractions until you see the terms repeating. (The terms you find may be checked against those given in Appendix B: Answers to Exercises.)

1. $\sqrt{11}$ 2. $\sqrt{56}$ 3. $\sqrt{39}$ 4. $\sqrt{79}$

Next, divide two of the larger convergents for each number getting a decimal value, and compare this with the values in the square root table in a mathematics handbook or extract the square root by a different method.

COMPARISON OF CONTINUED FRACTION REPRESENTATION WITH DECIMAL REPRESENTATION

It is interesting to compare the continued fraction representation of an irrational with the decimal expression of an irrational. We know that the digits in the decimal expression of an irrational number never repeat. We also know that if the digits of a decimal do repeat, then that decimal represents a rational number. (If you are not familiar with these ideas you may wish to refer to Proof No. 5 in Appendix A.) But you have just found that the terms of the continued fractions of four irrational numbers have nicely repeating terms. Did you notice that in each of your answers the last term before the beginning of each repeating set of terms is twice the first term? We shall prove later that this must always be true. What else can you find that is interesting about the terms of the continued fractions which you have developed?

USING THE THREE-STEP PROCESS TO EXPAND QUADRATIC IRRATIONALS

The three-step process can also be used to expand more general quadratic irrationals of the type $\frac{A + \sqrt{B}}{C}$ where A and C are integers different from zero, and where $\sqrt{B}$ is an irrational number.

EXAMPLE 2. Use the three-step process to develop the continued fraction for $\frac{1 + \sqrt{35}}{2}$.

Step 1.

$$\frac{1 + \sqrt{35}}{2} = 3 + \left(\frac{1 + \sqrt{35}}{2} - 3\right) = 3 + \frac{1 + \sqrt{35} - 6}{2} =$$

Step 2.

$$3 + \frac{-5 + \sqrt{35}}{2} = 3 + \cfrac{1}{\cfrac{2}{-5 + \sqrt{35}}} =$$

Step 3.

$$3 + \cfrac{1}{\cfrac{2}{(-5 + \sqrt{35})} \cdot \cfrac{(-5 - \sqrt{35})}{(-5 - \sqrt{35})}} = 3 + \cfrac{1}{\cfrac{2\,(-5 - \sqrt{35})}{25 - 35}} =$$

Step 1.

$$3 + \cfrac{1}{\cfrac{5 + \sqrt{35}}{5}} = 3 + \cfrac{1}{2 + \left(\cfrac{5 + \sqrt{35}}{5} - 2\right)} =$$

$$3 + \cfrac{1}{2 + \cfrac{5 + \sqrt{35} - 10}{5}} = 3 + \cfrac{1}{2 + \cfrac{-5 + \sqrt{35}}{5}} =$$

Step 2. *Step* 3.

$$3 + \cfrac{1}{2 + \cfrac{1}{\cfrac{5}{-5 + \sqrt{35}}}} = 3 + \cfrac{1}{2 + \cfrac{1}{\cfrac{5}{-5 + \sqrt{35}} \cdot \cfrac{(-5 - \sqrt{35})}{(-5 - \sqrt{35})}}} =$$

Step 1.

$$3+\cfrac{1}{2+\cfrac{1}{\cfrac{5+\sqrt{35}}{2}}} = 3+\cfrac{1}{2+\cfrac{1}{5+\left(\cfrac{5+\sqrt{35}}{2}-5\right)}} =$$

$$3+\cfrac{1}{2+\cfrac{1}{5+\cfrac{5+\sqrt{35}-10}{2}}} = 3+\cfrac{1}{2+\cfrac{1}{5+\cfrac{-5+\sqrt{35}}{2}}}$$

But we have already applied the process to $\frac{-5+\sqrt{35}}{2}$; so the terms are repeating, and the continued fraction for $\frac{+1+\sqrt{35}}{2}$ appears as follows:

$$\frac{+1+\sqrt{35}}{2} = 3+\cfrac{1}{2+\cfrac{1}{5+\cfrac{1}{2+\cfrac{1}{5+\cdots}}}}$$

It requires a considerable amount of effort to expand a quadratic irrational by the three-step process. However, since the ideas we are to explore stem from continued fractions that have been developed in this manner it is important that you *know how* to expand any quadratic irrational number by the three-step process. Later in this chapter we will develop an easier method for finding the terms of a quadratic. Now let us observe one more expansion.

EXAMPLE 3. Expand $\frac{2+\sqrt{5}}{3}$ into a continued fraction.

Step 1.

$$\frac{2+\sqrt{5}}{3} = 1+\frac{2+\sqrt{5}}{3}-1 = 1+\frac{2+\sqrt{5}-3}{3} = 1+\frac{-1+\sqrt{5}}{3} =$$

Step 2. *Step* 3.

$$1+\cfrac{1}{\cfrac{3}{-1+\sqrt{5}}} = 1+\cfrac{1}{\cfrac{3\,(-1-\sqrt{5})}{(-1+\sqrt{5})(-1-\sqrt{5})}} =$$

$$1+\cfrac{1}{\cfrac{3(-1-\sqrt{5})}{-4}} = 1+\cfrac{1}{\cfrac{3(1+\sqrt{5})}{4}}$$

But 3 is not an exact divisor of 4. This is the first case where we have encountered this problem. To handle this, multiply the numerator and the denominator of the original expression, $\frac{2+\sqrt{5}}{3}$, by 3, getting $\frac{6+\sqrt{45}}{9}$. Then expand this equivalent quadratic irrational, and all of the divisions will be exact. The first three steps are carried out here as an illustration. We start our expansion using $\frac{6+\sqrt{45}}{9}$ instead of $\frac{2+\sqrt{5}}{3}$:

$$\frac{6+\sqrt{45}}{9} = 1 + \left(\frac{6+\sqrt{45}}{9} - 1\right) = 1 + \frac{6+\sqrt{45}-9}{9} =$$

$$1 + \frac{-3+\sqrt{45}}{9} = 1 + \cfrac{1}{\cfrac{9}{-3+\sqrt{45}}} =$$

$$1 + \cfrac{1}{\cfrac{9 \qquad (-3-\sqrt{45})}{(-3+\sqrt{45})(-3-\sqrt{45})}} = 1 + \cfrac{1}{\cfrac{9(-3-\sqrt{45})}{-36}} =$$

$$1 + \cfrac{1}{\cfrac{3+\sqrt{45}}{4}}.$$

Exercise Set 15

Expand the following quadratic irrationals using the three-step process. Carry out the expansion until the terms start to repeat.

1. $\frac{1+\sqrt{11}}{2}$ 2. $\frac{3+\sqrt{15}}{2}$ 3. $\frac{6+\sqrt{2}}{2}$

Each of the quadratic irrationals that we have expanded into a continued fraction has resulted in a continued fraction with terms that repeat after a certain point. Do you feel that this would be true for every quadratic irrational? This question will be discussed again in a later chapter.

CONVERTING A REPEATING CONTINUED FRACTION INTO A QUADRATIC IRRATIONAL

We shall consider next the problem of converting a continued fraction which has repeating terms into a quadratic irrational. A method of doing this will now be illustrated by an example.

Example 4.

Convert $2 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{3 + \cdots}}}}}}$ to the form $\frac{A + \sqrt{B}}{C}$.

Solution. Now if we let X represent the continued fraction itself, and Y represent the repeating part, we get the following equations:

Equation 1.

$$X = 2 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{3 + \cdots}}}}}}$$

Equation 2.

$$Y = 5 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{Y}}}$$

Equation 3.

$$X = 2 + \frac{1}{Y}$$

Solving Equation 2 for Y, we find the following:

$$Y = 5 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{Y}}} = 5 + \cfrac{1}{1 + \cfrac{1}{\cfrac{3Y+1}{Y}}} = 5 + \cfrac{1}{1 + \cfrac{Y}{3Y+1}} =$$

$$5 + \cfrac{1}{\cfrac{3Y+1+Y}{3Y+1}} = 5 + \frac{3Y+1}{4Y+1} = \frac{20Y+5+3Y+1}{4Y+1} =$$

$$\frac{23Y+6}{4Y+1}.$$

We now have the equation $Y = \frac{23Y+6}{4Y+1}$.

This can be converted as follows:

$$4Y^2 + Y = 23Y + 6$$
$$4Y^2 - 22Y - 6 = 0$$
$$2Y^2 - 11Y - 3 = 0.$$

Y is a root of the quadratic equation $2Y^2 - 11Y - 3 = 0$. We also note, now, that Y is positive. Using the quadratic equation we can now write Y as follows:

$$Y = \frac{11 + \sqrt{11^2 - 4(2)(-3)}}{2 \cdot 2}$$

$$Y = \frac{11 + \sqrt{121 + 24}}{2 \cdot 2}$$

$$Y = \frac{11 + \sqrt{145}}{4}.$$

Substituting this value of Y in Equation 3, we can evaluate X, as follows:

$$X = 2 + \frac{1}{Y} = 2 + \cfrac{1}{\cfrac{11 + \sqrt{145}}{4}} = 2 + \frac{4}{11 + \sqrt{145}} =$$

$$2 + \frac{4}{(11 + \sqrt{145})}\frac{(11 - \sqrt{145})}{(11 - \sqrt{145})} = 2 + \frac{11 - \sqrt{145}}{-6} =$$

$$\frac{-12 + 11 - \sqrt{145}}{-6} = \frac{1 + \sqrt{145}}{6}.$$

The given repeating continued fraction can now be exhibited as a quadratic irrational number as required. Since $X = \frac{1 + \sqrt{145}}{6}$,

$$2 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{3} + \cdots}}}}} = \frac{1 + \sqrt{145}}{6}$$

Exercise Set 16

Use the method which was illustrated in Example 4 to convert the following repeating continued fractions to quadratic irrational numbers.

1. $2 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1} + \cdots}}}$

2. $3 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1} + \cdots}}}$

3. $$4 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{2 + \cdots}}}}$$

NON-TERMINATING CONTINUED FRACTIONS

We noted earlier that the continued fraction that corresponds to a rational number always terminates. You have noticed that the continued fractions which you have obtained for several irrational numbers never terminate but give rise to an infinite succession of terms. We shall now prove that this is true of all irrational numbers.

We start by investigating the expansion of an irrational number, X, into a continued fraction.

$$X = a_1 + X'$$

Here a_1 is the integer part of X.

$$X = a_1 + \frac{1}{\frac{1}{X'}}$$

Let

$$X_2 = \frac{1}{X'}.$$

$$X = a_1 + \frac{1}{X_2}$$

If X_2 is a rational number, we can now write

$$X = \frac{X_2 a_1 + 1}{X_2}.$$

This last equation implies that X is equal to a rational number, which is a contradiction since we assumed X to be irrational. We conclude, then, that X_2 is irrational.

The same argument could be used to show that X_3 is irrational, also X_4, X_5, etc. Our conclusion is that the continued fraction for any irrational number will not terminate.

TERM TABLES

Now that you have learned to expand quadratic irrationals into continued fractions, you are in a position to study some interesting relationships concerning the terms of these continued fractions. You will now learn a way to find the terms for the continued fraction of a quadratic irrational quickly and easily. This will be done through the use of *term tables*. Recall now our previous discussion: just where did we get the terms of continued fractions? Each term, you remember, was the integer part of some expression of the type $\frac{A + \sqrt{B}}{C}$. We shall now seek formulas that will give us the A's and C's more quickly than the three-step process.

We start by expanding the irrational $\frac{A + \sqrt{B}}{C}$. We note that $\frac{A + \sqrt{B}}{C} = a_1 + \frac{1}{u_2}$ where a_1 is the integer part of $\frac{A + \sqrt{B}}{C}$, and that u_2 is an irrational of the type $\frac{A_2 + \sqrt{B}}{C_2}$. We now try to discover a way of expressing u_3 which will be $\frac{A_3 + \sqrt{B}}{C_3}$ in terms of the integers involved in u_2, namely: A_2 and C_2 (B does not change).

We now apply the three-step process to u_2:

$u_2 = a_2 + \left(\frac{A_2 + \sqrt{B}}{C_2} - a_2\right)$, where a_2 is the integer part of u_2;

$$u_2 = a_2 + \frac{A_2 + \sqrt{B} - a_2C_2}{C_2} = a_2 + \frac{A_2 - a_2C_2 + \sqrt{B}}{C_2};$$

$$u_2 = a_2 + \frac{1}{\dfrac{C_2}{(A_2 - a_2C_2) + \sqrt{B}}} = a_2 + \frac{1}{\dfrac{C_2[(A_2 - a_2C_2) - \sqrt{B}]}{(A_2 - a_2C_2)^2 - B}} =$$

$$a_2 + \frac{1}{\dfrac{(A_2 - a_2C_2) - \sqrt{B}}{\dfrac{(A_2 - a_2C_2)^2 - B}{C_2}}} = a_2 + \frac{1}{\dfrac{(a_2C_2 - A_2) + \sqrt{B}}{\dfrac{B - (a_2C_2 - A_2)^2}{C_2}}}.$$

$$u_2 = \frac{1}{u_3} = a_2 + \frac{1}{\dfrac{A_3 + \sqrt{B}}{C_3}}.$$

Now by comparing these last expressions for u_2 with the previous three expressions for u_2, above, we get the desired formulas for A_3, and C_3:

$$A_3 = a_2C_2 - A_2$$

$$C_3 = \frac{B - A_3^2}{C_2}$$

$$u_3 = \frac{A_3 + \sqrt{B}}{C_3}$$

$$a_3 = \text{the integer part of } u_3.$$

Generalizing these formulas, we have the following:

$$A_n = a_{n-1}C_{n-1} - A_{n-1}$$

$$C_n = \frac{B - A_n^2}{C_{n-1}}$$

$$u_n = \frac{A_n + \sqrt{B}}{C_n}$$

$$a_n = \text{the integer part of } u_n.$$

The terms of the continued fraction for $\sqrt{71}$ will now be found by using the above formulas. Remember that each A, C, and a is found by using the preceding A, C, and a. Therefore we can get all of the a's (the terms) of the continued fraction that we desire just by knowing A_1, C_1, and a_1. These, of course, are just the A, the C, and the integer part of the original quadratic irrational. Let us now use these formulas in finding the terms for the continued fraction for $\sqrt{71}$.

Write: $\sqrt{71} = a_1 + \dfrac{A_1 + \sqrt{B}}{C_1} - a_1.$

But $\quad \sqrt{71} = 8 + \dfrac{0 + \sqrt{71}}{1} - 8$; so we have $A_1 = 0$, $C_1 = 1$, $a_1 = 8$.

Therefore, when using term tables to find the terms for a pure quadratic: A_1 is always 0, C_1 is always 1, and a_1 is the integer part of the given pure quadratic irrational.

Now place these values in a term table as shown in Table XVIII.

TABLE XVIII

n	1	2	3		. . .
A_n	0				
C_n	1				
a_n	8				

$$A_2 = a_1C_1 - A_1 = 8 \cdot 1 - 0 \cdot 8$$

$$C_2 = \frac{B - A_2^2}{C_1} = \frac{71 - 8^2}{1} = 7$$

$$u_2 = \frac{A_2 + \sqrt{B}}{C_2} = \frac{8 + \sqrt{71}}{7}$$

$$a_2 = \text{the integer part of } \frac{8 + \sqrt{71}}{7}\text{, which is 2.}$$

Don't try to memorize these formulas as you make your first term tables, but watch for the order in which you use the numbers which are already in the table. This will help you to see that with a little practice you can fill in the term tables without doing any "scratch" work on the side.

Table XIX is the term table for $\sqrt{71}$, completed for the first ten terms. You should practice with the formulas until you understand how the numbers in this table were obtained.

TABLE XIX

n	1	2	3	4	5	6	7	8	9	10
A_n	0	8	6	4	7	7	4	6	8	8
C_n	1	7	5	11	2	11	5	7	1	7
a_n	8	2	2	1	7	1	2	2	16	2

Notice that A_{10}, C_{10}, and a_{10} are the same as A_2, C_2, and a_2 respectively. Therefore a_{11} will be the same as a_3, a_{12} the same as a_4, etc. Thus the sequence of terms $a_2, a_3, \ldots, a_9$ will be exactly the same as the sequence $a_{10}, a_{11}, \ldots, a_{17}$, etc. So once more a pure quadratic has given us repeating terms in its continued fraction. Now, what have we discovered about the last term before the terms start to repeat? Is it twice a_1?

These formulas are quite general. A_1 and C_1 do not have to be positive.

EXAMPLE. Use a term table to find the terms of the continued fraction for $\dfrac{-2+\sqrt{7}}{-3}$.

$A_1 = -2 \qquad A_2 = a_1C_1 - A_1 = -1(-3) - (-2) = 3 + 2 = 5$

$C_1 = -3 \qquad C_2 = \dfrac{B - A_2^2}{C_1} = \dfrac{7-(5)^2}{-3} = \dfrac{7-25}{-3} = \dfrac{-18}{-3} = 6$

$a_1 = -1$

$u_2 = \dfrac{5+\sqrt{7}}{6}$

$a_2 = 1$

Table XX gives the term table for $\dfrac{-2+\sqrt{7}}{-3}$. The first five terms have been evaluated. The terms start repeating at $n = 5$.

TABLE XX

n	1	2	3	4	5
A_n	-2	5	1	2	1
C_n	-3	6	1	3	1
a_n	-1	1	3	1	3

Let us now try to make a term table for $\frac{2+\sqrt{7}}{2}$.

$A_1 = 2$

$B = 7$

$C_1 = 2$

$a_1 = 2$

n	1	2		...
A_n	2	2		
C_n	2			
a_n	2			

$$A_2 = a_1C_1 - A_1 = 2 \cdot 2 - 2 = 2$$

$$C_2 = \frac{B - A_2^2}{C_1} = \frac{7 - 2^2}{2} = \frac{7-4}{2} = \frac{3}{2}$$

But this last fraction is not an exact division! This happened once before when we tried to expand $\frac{2+\sqrt{5}}{3}$. In order to avoid this problem in the future, let us now prove a helpful theorem.

Theorem. *In the expansion of a given quadratic irrational* $\frac{A+\sqrt{B}}{C}$ *into a continued fraction,* $B - A_n^2$ *will always be exactly divisible by* C_{n-1} *if* $B - A^2$ *is exactly divisible by* C *and only if* $B - A^2$ *is exactly divisible by* C.

Before proving the theorem, note that A_1 and C_1 are respectively the A and C of the given quadratic irrational. Also remember that each C appearing in the continued fraction is $C_n = \frac{B - A_n^2}{C_{n-1}}$. Our method of proof will be to show that $B - (A_{n+1})^2$ is exactly divisible by C_n if and only if $B - A_n^2$ is exactly divisible by C_n.

Proof. By the formulas we use in constructing term tables, we find that:

$$C_{n+1} = \frac{B - A_{n+1}^2}{C_n} = \frac{B - (a_nC_n - A_n)^2}{C_n}$$

$$= \frac{B - (a_n^2C_n^2 - 2a_nC_nA_n + A_n^2)}{C_n}$$

$$= \frac{B - a_n^2C_n + 2a_nC_nA_n - A_n^2}{C_n}$$

$$= \frac{2a_nC_nA_n - a_n^2C_n + B - A_n^2}{C_n}$$

$$\frac{B - A_{n+1}^2}{C_n} = \frac{2a_nC_nA_n - a_n^2C_n^2}{C_n} + \frac{B - A_n^2}{C_n}.$$

γ

The numerator, $2a_nC_nA_n - a_n^2C_n^2$, in the last expression for $\frac{B - A_{n+1}^2}{C_n}$ is obviously divisible by C_n. We now have the desired result. Namely,

$B - A_{n+1}^2$ is divisible by C_n if and only if $B - A_n^2$ is divisible by C, This means that $B - A_2^2$ is divisible by C_1 if and only if $B - A_1^2$ i divisible by C_1. Therefore you should not start expanding a quadrati of the form $\frac{A + \sqrt{B}}{C}$ into a continued fraction, and you should no start to construct a term table until you have checked to see tha $B - A^2$ is divisible by C. Then if it is not, multiply the numerator anc denominator of $\frac{B - \sqrt{A}}{C}$ by C as follows:

$$\frac{C(B - \sqrt{A})}{C \cdot C} = \frac{BC - \sqrt{AC^2}}{C^2}.$$

And now it is easy to see that $AC^2 - (BC)^2$ *is* divisible by C^2.

In the problem of expanding $\frac{2 + \sqrt{7}}{2}$ into a continued fraction: $A_1 = 2, B = 7$, and $C_1 = 2$. We know that $B_1 - A_1^2$ (which is $7 - 2^2$) is not divisible by C_1 (which is 2). Therefore multiply numerator and denominator by 2, getting $\frac{4 + \sqrt{28}}{4}$. Now you can expand this into a continued fraction, and all divisions will be exact.

Exercise Set 17

Make term tables for the following quadratic irrational numbers. Carry each table out until the terms start to repeat.

1. $\sqrt{47}$
2. $\sqrt{22}$
3. $\sqrt{34}$
4. $\frac{3 + \sqrt{39}}{2}$
5. $\frac{2 + \sqrt{13}}{3}$
6. $\frac{-3 + \sqrt{21}}{4}$
7. $\frac{-3 + \sqrt{35}}{-2}$
8. $\frac{2 + \sqrt{6}}{4}$

CHAPTER 8

CONTINUED FRACTIONS AND PELL'S EQUATION

PRELIMINARY INVESTIGATION

Before starting a study of Pell's equation, you should re-examine the terms of each of the continued fractions you have found so far for pure quadratic irrationals (numbers of the type $\sqrt{B}$). You should find that the terms of each of these continued fractions form a sequence of the following type:

$$a_1, a_2, a_3, \cdots, a_n, 2a_1, a_2, a_3, \cdots, a_n, 2a_1, a_2, \cdots$$

EXAMPLES.

For $\sqrt{14}$ the terms are 3, 1, 2, 1, 6, 1, 2, 1, 6, ⋯

For $\sqrt{23}$ the terms are 4, 1, 3, 1, 8, 1, 3, 1, 8, ⋯

For $\sqrt{47}$ the terms are 6, 1, 5, 1, 12, 1, 5, 1, 12, ⋯

You should find that the terms in your term tables for pure quadratic irrationals also have the property that the repeating series starts with a_2, and the last term in the repeating series is $2a_1$. We shall later prove that this must always be true for the terms of pure quadratic irrationals.

PELL'S EQUATION

An equation of the form $x^2 - Py^2 = 1$, where P is a positive integer, is called a *Pell's equation.* We shall now show that integral values of x and y that will satisfy any equation of this type can always be found.

For example, $x^2 - 39y^2 = 1$ is a Pell's equation; and $x = 25$, and $y = 4$ are solutions for this equation.

Check: $x^2 - 39y^2 = 1$

$$25^2 - 39 \cdot 4^2 = ?$$
$$625 - 39 \cdot 16 = ?$$
$$625 - 624 = 1$$

Let us investigate the continued fraction for $\sqrt{P}$ to see if we can discover a relationship between it and Pell's equation, $x^2 - Py^2 = 1$.

$$\sqrt{P} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + \cfrac{1}{a_n + \cfrac{1}{2a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}}}}$$

$$\sqrt{P} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + \cfrac{1}{a_n + \cfrac{1}{a_1 + \underbrace{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}_{\sqrt{P}}}}}}$$

$$\sqrt{P} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + \cfrac{1}{a_n + \cfrac{1}{a_1 + \sqrt{P}}}}}$$

In the last expression for $\sqrt{P}$ the last term, a_{n+1}, is $a_1 + \sqrt{P}$. Now since the last convergent $\left(\frac{r_{n+1}}{s_{n+1}} \text{ in this case}\right)$ is equal to the number the continued fraction represents, we apply the formula for the $(n + 1)$th convergent, getting:

$$C_{n+1} = \frac{r_{n+1}}{s_{n+1}} = \sqrt{P} = \frac{a_{n+1}r_n + r_{n-1}}{a_{n+1}s_n + s_{n-1}}.$$

But $$a_{n+1} \text{ is } a_1 + \sqrt{P}.$$

$$\sqrt{P} = \frac{(a_1 + \sqrt{P})r_n + r_{n-1}}{(a_1 + \sqrt{P})s_n + s_{n-1}}$$

$$\sqrt{P}[(a_1 + \sqrt{P})s_n + s_{n-1}] = (a_1 + \sqrt{P})r_n + r_{n-1}$$

$$\sqrt{P}s_n(a_1 + \sqrt{P}) + \sqrt{P}s_{n-1} = a_1r_n + r_n\sqrt{P} + r_{n-1}$$

$$a_1s_n\sqrt{P} + s_nP + s_{n-1}\sqrt{P} = a_1r_n + r_n\sqrt{P} + r_{n-1}$$

$$s_nP + (a_1s_n + s_{n-1})\sqrt{P} = a_1r_n + r_{n-1} + r_n\sqrt{P}$$

The left and right members of this last equation are equal. But a rational number cannot equal an irrational number. Therefore the rational parts must be equal, and the irrational parts must be equal. As a result of this observation, we have the following equations:

$$s_nP = a_1r_n + r_{n-1} \quad \text{and} \quad a_1s_n + s_{n-1} = r_n$$

or $$r_{n-1} = s_nP - a_1r_n \quad \text{and} \quad s_{n-1} = r_n - a_1s_n.$$

But from our discussion of the crisscross products in the convergent tables we have

$$r_ns_{n-1} - s_nr_{n-1} = (-1)^n.$$

Now substituting in this equation the values for r_{n-1} and s_{n-1} from the two previous equations we get the following:

$$r_n(r_n - a_1s_n) - s_n(s_nP - a_1r_n) = (-1)^n$$

$$r_n^2 - a_1r_ns_n - Ps_n^2 + a_1r_ns_n = (-1)^n$$

$$r_n^2 - Ps_n^2 = (-1)^n.$$

We see that r_n and s_n are solutions to the equation $x^2 - Py^2 = (-1)^n$. Thus, if we want integers which will satisfy an equation of the type $x^2 - Py^2 = (-1)^n$, we find the terms for the continued fraction for $\sqrt{P}$ and form a table of convergents. If n is the number of terms before the term $2a_1$ appears; then, $x = r_n$ and $y = s_n$, which are the numerator and denominator respectively of the nth convergent, are solutions for the equation. If n is even, $x = r_n$ and $y = s_n$ are solutions for $x^2 - Py^2 = 1$. If n is odd, we have solutions for $x^2 - Py^2 = -1$. If we insist upon solutions for $x^2 - Py^2 = 1$ and if n is odd, we then use as our solutions $x = r_{2n}$ and $y = s_{2n}$; and since $2n$ is an even number, we have

$$(r_{2n})^2 - P(s_{2n})^2 = 1.$$

SOLUTIONS FOR TWO PELL'S EQUATIONS

The theory will now be illustrated by solving two Pell's equations, one with n an even number and one with n an odd number.

EXAMPLE 1. Find integral solutions for the equation:

$$x^2 - 28y^2 = 1.$$

First form the term table and then the convergent table for $\sqrt{28}$.

TERM TABLE

n	1	2	3	4	5
A_n	0	5	4	4	5
C_n	1	3	4	3	1
a_n	5	3	2	3	10

CONVERGENT TABLE

n	−1	0	1	2	3	4
a_n			5	3	2	3
r_n	0	1	5	16	37	127
s_n	1	0	1	3	7	24

For this example we use $n = 4$, because for $n = 5$ the term $a_5 = 10$ which is twice a_1. So the numerator and denominator of the fourth convergent are the required values for x and y. We have $x = 127$ and $y = 24$.

Check: $x^2 - 28y^2 = 1$ $\quad 127^2 - 28 \cdot 24^2 = ?$

$$16{,}129 - 16{,}128 = 1$$

It was stated that if n is odd, then r_{2n} and s_{2n} would be solutions to $x^2 - Py^2 = 1$. It is true that $2n$ is even, but this in itself does not mean that r_{2n} and s_{2n} are solutions. However, if you will review in this chapter our initial investigation of the form of the continued fraction for $\sqrt{P}$, you will see that $a_n = a_{2n}$, $a_{n+1} = a_{2n+1}$, and $a_{n-1} = a_{2n-1}$. We could carry through the same work as before for C_{2n+1} using $a_{2n+1} = a_{2n} + \sqrt{P}$ and get the result $r_{2n}^2 - Ps_{2n}^2 = (-1)^{2n}$; and since $2n$ is even we would have

$$r_{2n}^2 - Ps_{2n}^2 = 1.$$

Let us now apply these ideas.

EXAMPLE 2. Find integers that satisfy the equation

$$x^2 - 41y^2 = 1.$$

TERM TABLE FOR $\sqrt{41}$

n	1	2	3	4
A_n	0	6	4	6
C_n	1	5	5	1
a_n	6	2	2	12

Here the term 12, which is $2a_1$, is a_4; so $n = 3$. But if n is odd, the solutions to our equation will be r_{2n} and s_{2n}. In this case we will want r_6 and s_6. Since the terms repeat, it is not necessary to calculate more

a's. We know $a_5 = 2$, and $a_6 = 2$. Now placing these in a convergent table, as shown in Table XXI, we find r_6 and s_6.

TABLE XXI

n	-1	0	1	2	3	4	5	6
a_n			6	2	2	12	2	2
r_n	0	1	6	13	32	397	826	2,049
s_n	1	0	1	2	5	62	129	320

$$x = r_6 = 2{,}049$$

$$y = s_6 = 320$$

Check: $x^2 - 41y^2 = 1 \qquad 2049^2 - 41(320)^2 = ?$

$$4{,}198{,}401 - 4{,}198{,}400 = 1$$

Exercise Set 18

Find integral solutions for the following Pell's equations.

1. $x^2 - 7y^2 = 1$
2. $x^2 - 21y^2 = 1$
3. $x^2 - 34y^2 = 1$
4. $x^2 - 13y^2 = 1$
5. $x^2 - 29y^2 = 1$

Regardless of whether n is even or odd, we know $2n$ is even. Now since $x = r_{2n}$ and $y = s_{2n}$ are solutions to the equation $x^2 + Py^2 = 1$, we have here a method of obtaining more solutions to any equation of the type $x^2 + Py^2 = 1$.

If n is even, then $x = r_n$ and $y = s_n$ satisfy the equation. Another pair of integers which will satisfy the same equation is r_{2n} and s_{2n}. In general, solutions are $x = r_{kn}$ and $y = s_{kn}$ for any positive integer k. If n is odd, then you must use $x = r_{2n}$ and $y = s_{2n}$ for your first pair of solutions. Therefore, since the product of 3 and any odd number is an odd number, you must use $x = r_{4n}$ and $y = s_{4n}$ for your second pair of solutions. In general, if n is odd, you will have solutions to the Pell's equation $x^2 + Py^2 = 1$ by using $x = r_{bn}$ and $y = s_{bn}$ where b is an *even* positive integer.

EXAMPLE 3. Find two pairs of integers which will satisfy the equation $x^2 - 39y^2 = 1$.

TERM TABLE FOR $\sqrt{39}$

n	1	2	3	4
A_n	0	6	6	6
C_n	1	3	1	3
a_n	6	4	12	4

Here the term 12, which is $2a_1$, is a_3; so we use $n = 2$ and find that $x_1 = r_2$ and $y_1 = s_2$ are solutions. Our discussion above tells us that $x_2 = r_4$ and $y_2 = s_4$ should be another pair of solutions.

Let us now construct a convergent table for the first four convergents.

CONVERGENT TABLE FOR $\sqrt{39}$

n	-1	0	1	2	3	4
a_n			6	4	12	4
r_n	0	1	6	25	306	1,249
s_n	1	0	1	4	49	200

$x_1 = r_2 = 25$ $\qquad x_2 = r_4 = 1{,}249$

$y_1 = s_2 = 4$ $\qquad y_2 = s_4 = 200$

Check: $x^2 - 39y^2 = 1$ $\qquad$ *Check*: $x^2 - 39y^2 = 1$

$$25^2 - 39 \cdot 4^2 = ? \qquad 1249^2 - 39 \cdot 200^2 = ?$$

$$625 - 624 = 1 \qquad 1{,}560{,}001 - 1{,}560{,}000 = 1$$

Exercise Set 19

Using the ideas just discussed, find the next pair of solutions to the following equations (these equations are the same as the first three equations in Exercise Set 18).

1. $x^2 - 7y^2 = 1$ **2.** $x^2 - 21y^2 = 1$ **3.** $x^2 - 34y^2 = 1$

We have shown that r_n and s_n are integral solutions of the equation $x^2 - Py^2 = 1$ when n is the number of the term preceding the term $2a_1$ in the continued fraction for P. We also showed that r_{2n} and s_{2n} are solutions. Now if n is very large, say 6, the process of obtaining all of the convergents from $\frac{r_7}{s_7}$ to $\frac{r_{12}}{s_{12}}$ involves many arithmetic computations with numbers that are probably very large. It would, therefore, be convenient if we could discover formulas that would enable us to find r_{2n} and s_{2n} in terms of r_n and s_n directly, without having to evaluate all of the r's and s's in between.

Let us start looking for such formulas by examining the case where $n = 2$. Assume that r_2 and s_2 are the solutions to some Pell's equation, $x^2 - Py^2 = 1$. If this is true, $r_2 = x$ and $s_2 = y$. We will then use these values to compute r_4 and s_4 by means of a convergent table. If r_2 and s_2 are solutions, then r_4 and s_4 are also. After finding an expression for r_4 and s_4, we will attempt to express r_4 and s_4 in terms of x, y, and P.

TABLE XXII

n	-1	0	1	2	3	4
a_n			a_1	a_2	$2a_1$	a_2
r_n	0	1	a_1	$a_1a_2 + 1 = x$	$2a_1x + a_1$	$2a_1a_2x + a_1a_2 + x$
s_n	1	0	1	$a_2 = y$	$2a_1y + 1$	$2a_1a_2y + a_2 + y$

$$
\begin{aligned}
s_4 &= 2a_1a_2y + y && + a_2 \\
&= a_1a_2y + a_2 && + a_1a_2y + y && \text{but } a_2 = y \\
&= a_1a_2y + y && + y(a_1a_2 + 1) \\
&= y(a_1a_2 + 1) && + y(a_1a_2 + 1) && \text{but } a_1a_2 + 1 = x \\
&= yx && + yx \\
s_4 &= 2xy \\
r_4 &= 2a_1a_2x + a_1a_2 + x \\
&= a_1a_2x + x && + a_1a_2x + a_1a_2 \\
&= x(a_1a_2 + 1) && + a_2(a_1x + a_1) && \text{but } x = a_1a_2 + 1 \text{ and } a_2 = y \\
&= x \cdot x && + y(a_1x + a_1)
\end{aligned}
$$

Now multiply and divide $y(a_1x + a_1)$ by y.

$$r_4 = x^2 + y^2 \left(\frac{a_1x + a_1}{y}\right)$$

Note now that any solution to $x^2 - Py^2 = 1$ is determined by P; therefore, P must appear somewhere in our expressions for r_4 and s_4. Is it possible that $\frac{a_1x + a_1}{y}$ is equal to P? Solving $x^2 - Py^2 = 1$ for P gives:

$$
\begin{aligned}
x^2 - Py^2 &= 1 \\
Py^2 &= x^2 - 1 \\
P &= \frac{x^2 - 1}{y^2}.
\end{aligned}
$$

Now $a_2 = y$, so multiply the numerator of $\frac{a_1x + a_1}{y}$ by a_2 and the denominator by y.

$$r_4 = x^2 + y^2 \left(\frac{a_1a_2x_1 + a_1a_2}{y^2}\right)$$

The numerator of the expression for P involves -1, and the denominator is y^2. The expression which we think might be equal to P has now a denominator of y^2, which is what we want; so instead of performing a division to get -1 in the numerator, let us add $+1$ and -1 to the numerator:

$$r_4 = x^2 + y^2\left(\frac{a_1a_2x + a_1a_2 + 1 - 1}{y^2}\right).$$

Now substitute x for $a_1a_2 + 1$:

$$= x^2 + y^2\left(\frac{a_1a_2x + x - 1}{y^2}\right)$$

$$= x^2 + y^2\left(\frac{x(a_1a_2 + 1) - 1}{y^2}\right).$$

Again substitute x for $a_1a_2 + 1$:

$$= x^2 + y^2\left(\frac{x \cdot x - 1}{y^2}\right) = x^2 + y^2\left(\frac{x^2 - 1}{y^2}\right).$$

But we saw before that

$$P = \frac{x^2 - 1}{y^2}.$$

Substitution gives us

$$r_4 = x^2 + y^2P$$

and this is the kind of expression that we have been trying to find.

We now know that if $x = r_2$ and $y = s_2$ are solutions to $x^2 - Py^2 = 1$, then the solutions r_4 and s_4 are given by the formulas:

$$r_4 = r^2_2 + Ps^2_2 \qquad \text{and} \qquad s_4 = 2r_2s_2.$$

This suggests that in general: if x_1 and y_1 are solutions to $x^2 - Py^2 = 1$, then $x_2 = x_1^2 + Py_1^2$ and $y_2 = 2x_1y_1$ are also solutions. It is not difficult to prove that these formulas always hold true. (Proof is given in Appendix A, Proof No. 4.)

The following example illustrates the use of these formulas.

EXAMPLE. $x_1 = 161$ and $y_1 = 24$ are solutions to the equation

$$x^2 - 45y^2 = 1.$$

Find another pair of integers x_2 and y_2 which will satisfy this equation.

$$x_2 = x_1^2 + Py_1^2 \qquad y_2 = 2x_1y_1$$

$$x_2 = 161^2 + 45 \cdot 24^2 \qquad y_2 = 2(161)24$$

$$x_2 = 25{,}921 + 25{,}920 \qquad y_2 = 7{,}728$$

$$x_2 = 51{,}841$$

Check:

$$x_2^2 - Py_2^2 = 1 \qquad 51{,}841^2 - 45(59{,}721{,}984) = ?$$

$$2{,}687{,}489{,}281 - 2{,}687{,}489{,}280 = 1$$

Exercise Set 19a

The following five equations of the type $x^2 - Py^2 = 1$ are presented with one pair of solutions for each equation. Using the formulas just developed, find a second pair of solutions and check by substituting into the original equation.

1. $x^2 - 6y^2 = 1 \qquad x_1 = 5,\ y_1 = 2$
2. $x^2 - 12y^2 = 1 \qquad x_1 = 7,\ y_1 = 2$
3. $x^2 - 26y^2 = 1 \qquad x_1 = 5,\ y_1 = 1$
4. $x^2 - 38y^2 = 1 \qquad x_1 = 37,\ y_1 = 6$

CHAPTER 9

INITIALLY REPEATING CONTINUED FRACTIONS AND QUADRATIC EQUATIONS

TERMINOLOGY EMPLOYED

At this time let us recall some of the terminology used in discussing quadratic irrationals. Once again, by a *quadratic irrational* we mean a number of the type $\frac{A + \sqrt{B}}{C}$ where A and C are integers, and $C \neq 0$, and B is a positive integer such that $\sqrt{B}$ is an irrational number.

Every quadratic irrational has a *conjugate*. The conjugate is the same number with the sign of the irrational part changed. Following are some examples:

1. The conjugate of $3 + \sqrt{5}$ is $3 - \sqrt{5}$.
2. The conjugate of $-\sqrt{7}$ is $\sqrt{7}$.
3. The conjugate of $\frac{2 + \sqrt{7}}{5}$ is $\frac{2 - \sqrt{7}}{5}$.

If one root of a quadratic equation is irrational then the second root is the *conjugate of the first root*. For example, the roots of $x^2 - 4x - 1 = 0$ are $x_1 = 2 + \sqrt{5}$ and $x_2 = 2 - \sqrt{5}$.

Now look at a repeating continued fraction in which the repeating sequence starts with a_1. We shall call continued fractions of this type *initially repeating continued fractions*. In general, they are of the following form.

$$a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_n + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_n + \cdots}}}}}$$

If X stands for the number this continued fraction represents, we can then write:

$$X = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + \cfrac{1}{a_n + \cfrac{1}{X}}}}$$

Now in this expression X, itself, takes the place of a_{n+1}, so applying the formula for the ($n+1$th) convergent we have the following:

$$C_{n+1} = \frac{r_{n+1}}{s_{n+1}} = X = \frac{Xr_n + r_{n-1}}{Xs_n + s_{n-1}}$$

$$s_1X^2 + s_{n-1}X = r_nX + r_{n-1}$$

$$s_nX^2 + (s_{n-1} - r_n)X - r_{n-1} = 0.$$

This last equation will be called the *quadratic equation of the initially repeating continued fraction*, X. Now what can we say about the roots of this equation? We know that X (the value of the continued fraction) is positive. Therefore we know the equation has one positive root. But what do we know about the other root? Since it was indicated earlier that a particular continued fraction can represent only one number, the other root must be either negative or equal to X. The roots of a quadratic equation, $ax^2 + bx + c = 0$, are

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

If they are equal, we have

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$\text{or} \qquad +\sqrt{b^2 - 4ac} = -\sqrt{b^2 - 4ac}.$$

This last statement of equality can only be true if $b^2 - 4ac = 0$. From the quadratic equation for our initially repeating continued fraction,

$$s_n X^2 + (s_{n-1} - r_n)X - r_{n-1} = 0,$$

$$a = s_n, \qquad b = s_{n-1} - r_n, \qquad \text{and} \qquad c = -r_{n-1}.$$

$$b^2 - 4ac = (s_{n-1} - r_n)^2 - 4(s_n)(-r_{n-1})$$

$$b^2 - 4ac = (s_{n-1} - r_n)^2 + 4s_n r_{n-1}$$

Now r_n, r_{n-1}, s_n, and s_{n-1} are positive integers; so $(s_{n-1} - r_n)^2 + 4s_n r_{n-1}$ cannot possibly equal zero. We have, therefore, proved the following:

> THEOREM 6. *The quadratic equation for an initially repeating continued fraction always has one positive and one negative root.*

FINDING THE QUADRATIC EQUATION FOR A CONTINUED FRACTION

We now find the quadratic equation for the following continued fraction, X.

$$X = 3 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{4} + \cdots}}}}}}$$

$$X = 3 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{X}}}} = 3 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{\frac{4X + 1}{X}}}} =$$

$$3 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{X}{4X + 1}}} = 3 + \cfrac{1}{1 + \cfrac{1}{\frac{8X + 2 + X}{4X + 1}}} =$$

$$3 + \cfrac{1}{\cfrac{1 + 4X + 1}{9X + 2}} = 3 + \cfrac{1}{\cfrac{9X + 2 + 4X + 1}{9X + 2}} = 3 + \frac{9X + 2}{13X + 3} =$$

$$\frac{39X + 9 + 9X + 2}{13X + 3} = X = \frac{48X + 11}{13X + 3}$$

$13X^2 - 45X - 11 = 0$, which is the desired equation.

The quadratic equation for an initially repeating fraction can be found much more easily by employing a convergent table. When we consider the fifth term as being X, as above, the continued fraction has five terms. Therefore X is equal to the fifth convergent for this continued fraction. Set up a table showing the five convergents (Table XXIII).

Table XXIII

n	-1	0	1	2	3	4	5
a_n			3	1	2	4	X
r_n	0	1	3	4	11	48	$48X + 11$
s_n	1	0	1	1	3	13	$13X + 3$

$$C_5 = \frac{r_5}{s_5} = X = \frac{48X + 11}{13X + 3}$$

$$13X^2 - 45X - 11 = 0$$

Compare the structures of the following two equations:

Eq. 1. $3x^2 + 4x - 2 = 0$ Eq. 2. $2z^2 + 4z - 3 = 0$

Equation 2 is formed by reversing the order of the natural numbers which appear in the coefficients of Equation 1. The sign of each term is left unchanged. Now what is the relation between the positive roots of two quadratic equations constructed in this way? Let us start to look for an answer by first finding these roots, as follows:

$$x = \frac{-4 + \sqrt{4^2 - 4 \cdot 3(-2)}}{2 \cdot 3} \qquad z = \frac{-4 + \sqrt{4^2 - 4 \cdot 2(-3)}}{2 \cdot 2}$$

$$x = \frac{-2 + \sqrt{10}}{3} \qquad z = \frac{-2 + \sqrt{10}}{2}$$

Now, let x' be the conjugate of x.

$$x' = \frac{-2 - \sqrt{10}}{}$$

Consider the following:

$$-\frac{1}{z} = -\frac{1}{\frac{-2+\sqrt{10}}{2}}$$

$$= -\frac{2}{-2+\sqrt{10}}$$

$$= \frac{-2}{(-2+\sqrt{10})}\,\frac{(-2-\sqrt{10})}{(-2-\sqrt{10})}$$

$$\frac{-2(-2-\sqrt{10})}{4-10}$$

$$-\frac{1}{z} = \frac{-2-\sqrt{10}}{3}.$$

We have $x' = -\frac{1}{z}$. Thus if two quadratic equations are constructed in the same manner as the two above, and x is a positive root of one equation, and z is a positive root of the other, then, $x' = -\frac{1}{z}$. This is always true, but no further proof will be given here.

At this point it seems reasonable to ask, "What is the relationship between a given initially repeating continued fraction and the continued fraction formed by reversing the order of the repeating terms?" For example, what is the relationship between the following two continued fractions?

$$2+\cfrac{1}{1+\cfrac{1}{5+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{5+\dots}}}}} \quad \text{and} \quad 5+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{5+\cfrac{1}{1+\cfrac{1}{2+\dots}}}}}$$

Set the first equal to y and the second equal to z.

$$y = 2+\cfrac{1}{1+\cfrac{1}{5+\cfrac{1}{y}}} \qquad z = 5+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{z}}}$$

Make convergent tables for the convergents of both y and z.

TABLE XXIV

n	-1	0	1	2	3	4
a_n			2	1	5	y
r_n	0	1	2	3	17	$17y + 3$
s_n	1	0	1	1	6	$6y + 1$

TABLE XXV

n	-1	0	1	2	3	4
a_n			5	1	2	z
r_n	0	1	5	6	17	$17z + 6$
s_n	1	0	1	1	3	$3z + 1$

$$y = \frac{17y + 3}{6y + 1} \qquad z = \frac{17z + 6}{3z + 1}$$

$$6y^2 + y = 17y + 3 \qquad 3z^2 + z = 17z + 6$$

$$6y^2 - 16y - 3 = 0 \qquad 3z^2 - 16z - 6 = 0$$

Note that these two quadratic equations are constructed in the same manner as those that we have been discussing. Therefore $y' = -\frac{1}{z}$. Actually $y = \frac{8 + \sqrt{82}}{6}$, and $z = \frac{8 + \sqrt{82}}{3}$. You should check the relationship.

We see that if X is any initially repeating continued fraction and Y is the continued fraction formed by reversing the terms of the repeating sequence, the following statements are true.

1. X and Y are both greater than 1 because a_1 in either case is a positive integer.
2. Since Y is greater than 1, $\frac{1}{Y}$ is less than 1. Therefore $-\frac{1}{Y}$ is negative but greater than -1.
3. X' (the conjugate of X) is equal to $-\frac{1}{Y}$. Therefore X' is negative but greater than -1.

As a result of the three statements above, we can say that the following inequalities, or properties, exist.

(a) $X > 1$

(b) $-1 < X' < 0$

Any quadratic irrational X which possesses the two properties (a) and (b) is called a *reduced* quadratic irrational.

As a consequence of the observations made in this chapter, we now state:

THEOREM 7. *Every initially repeating continued fraction represents a reduced quadratic irrational.*

Exercise Set 20

Rewrite the following initially repeating continued fractions as quadratic irrationals by finding the quadratic equation for each continued fraction and solving for the positive root. Then check to see that this root is a reduced quadratic irrational.

1. $2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cdots}}}$

2. $3 + \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{4 + \cdots}}}}}$

Historical Note

The relationships between a repeating continued fraction and the continued fraction formed by writing the repeating sequence of terms in reverse order were studied extensively by the young French mathematician, Evariste Galois (pronounced galwah). He was born in 1811 and died in 1832. (Note his age.) Galois made important contributions to the field of mathematics. He was the first to prove that a fifth-degree equation cannot, in general, be solved by ordinary algebra. He also showed exactly which equations are solvable. His investigations are basic to the theory of groups which is extremely important to modern-day mathematicians.

It is amazing that Galois accomplished all of this before he was twenty-one years of age. He was killed in a duel when he was twenty years old. If you would like to read more of the details of the interesting and exciting life of Evariste Galois, you should read *Whom the Gods Love* by Leopold Infeld. New York: Whittlesey House, 1948. If you wish to read a clear explanation of his theory of groups, you may read *Galois and the Theory of Groups* by Lillian R. Lieber and Hugh Gray Lieber. Lancaster Pennsylvania: Science Press Printing Co., 1932.

CHAPTER 10

INITIALLY REPEATING CONTINUED FRACTIONS AND REDUCED QUADRATIC IRRATIONALS

AN INTERESTING QUESTION

We saw in Chapter 9 that every initially repeating continued fraction represents a reduced quadratic irrational. Is it true also that the continued fraction of every reduced quadratic is initially repeating? Let us begin our investigation by asking: "Does the continued fraction of a reduced quadratic ever repeat in any manner?" Our next step is to study the structure of a reduced quadratic.

Let our reduced quadratic irrational be R, and its conjugate be R'. Now r is a root of some quadratic equation

$$aR^2 + bR + c = 0$$

where a, b, and c are integers. Applying the quadratic formula we get

$$R = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Notice that here $-b$ is an integer, $b^2 - 4ac$ is an integer, and $2a$ is an integer. Since R is of the form $\dfrac{A \pm \sqrt{B}}{C}$, we have $A = -b$, $B = b^2 - 4ac$, and $C = 2a$; and this tells us, further, that A, B, and C are integers. If the sign before $\sqrt{B}$ is not $+$, we can make it so by multiplying both numerator and denominator of the quadratic irrational by -1. We now assume R and R' to be of the form

$$R = \frac{A + \sqrt{B}}{C} \qquad \text{and} \qquad R' = \frac{A - \sqrt{B}}{C}.$$

We now use the rest of the properties of the reduced quadratic R:

$$R' < 0 \qquad \text{means} \qquad \frac{-b - \sqrt{b^2 - 4ac}}{2a} < 0$$

$$\text{or} \qquad \frac{A - \sqrt{B}}{C} < 0.$$

Now multiply both sides of this inequality by C, getting

$$A - \sqrt{B} \quad < 0$$

$$\text{or} \quad A < \sqrt{B}.$$

$$R > 1 \qquad \text{means} \qquad \frac{-b + \sqrt{b^2 - 4ac}}{2a} > 1$$

$$\text{or} \qquad \frac{A + \sqrt{B}}{C} > 1;$$

and multiplying by C, we get $A + \sqrt{B} > C$.

$$\text{But} \qquad A < \sqrt{B};$$

adding $\sqrt{B}$ to each side, we get $A + \sqrt{B} < 2\sqrt{B}$. And since $A + \sqrt{B} > C$, we have

$$2\sqrt{B} > A + \sqrt{B} > C$$

$$\text{or} \qquad C < 2\sqrt{B}.$$

Recall now that when we constructed term tables for the terms of the continued fraction for a quadratic irrational, we employed the expression

$$\frac{B - (A_n)^2}{C_{n-1}}$$

and found that $B - (A_n)^2$ was always exactly divisible by C_{n-1}. Perhaps this can lead us to another relationship between the A, B, and C in our reduced quadratic.

We want to discover all that we can about the integers A, B, and C which are involved in a reduced quadratic. Now since R is reduced, we have $R > 1$ and $-1 < R' < 0$; therefore, $R - R' > 0$, and $R + R' > 0$. This means

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} - \frac{-b - \sqrt{b^2 - 4ac}}{2a} > 0,$$

$$\text{or} \quad \frac{2\sqrt{b^2 - 4ac}}{2a} > 0.$$

But keep in mind that it is the relationships between the integers A, B, and C that we are trying to find. Dividing both sides of the inequality 2 we get

$$\frac{\sqrt{b^2 - 4ac}}{2a} > 0$$

$$\frac{\sqrt{B}}{C} > 0$$

and now, since $\sqrt{B}$ is positive, it follows that C is positive.

$$R + R' > 0 \quad \text{means} \quad \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} > 0$$

$$\frac{-2b}{2a} > 0$$

and since $A = -b$, and $C = 2a$, we divide both sides of the inequality by 2 getting

$$\frac{-b}{2a} > 0$$

$$\text{or} \quad \frac{A}{C} > 0$$

and since C is positive, this shows that A is positive.

We now have found the following: A is positive, B is positive, and C is positive. So, if a quadratic irrational is reduced, all signs involved are $+$.

$$\frac{B - A^2}{C} = \frac{b^2 - 4ac - (-b)^2}{2a}$$

$$= \frac{b^2 - 4ac - b^2}{2a}$$

$$= \frac{-4ac}{2a}$$

$$= -2c$$

Now $-2c$ is an integer, so this shows that $B - A^2$ is always divisible by C. And as we pointed out in our previous discussion of this property: if $B - A^2$ is not divisible by C, just multiply both numerator and denominator of the quadratic irrational by C.

We now have four restrictions upon the integers A, B, and C of a reduced quadratic irrational $\frac{A + \sqrt{B}}{C}$. They are as follows:

$$(1) \qquad A, C, \text{ and } \sqrt{B} \text{ are positive.}$$

$$(2) \qquad A < \sqrt{B}.$$

$$(3) \qquad C < 2\sqrt{B}.$$

$$(4) \qquad B - A^2 \text{ is divisible by } C.$$

Now what does this mean? Let us make a reduced quadratic with $B = 5$. What are the possibilities for A? By (1) and (2), A can only be 1 or 2. Then what are the possibilities for C? By (3), C can only be 1, 2, 3, or 4; and by (4), $B - A^2$ must be divisible by C. So if A is 1, C can be 1, 2, or 4; and if A is 2, C can only be 1.

The point is that for any value of B there are only a limited number of values of A and C that can make $\frac{A + \sqrt{B}}{C}$ a reduced quadratic.

We need to show now that each u_n of the form $\frac{A_n + \sqrt{B}}{C_n}$ which occurs in the expansion of the reduced quadratic $\frac{A + \sqrt{B}}{C}$ is itself a reduced quadratic. Remember that we showed previously that the continued fraction for any irrational number never terminates. What do you think our conclusion will be if each of the u's is shown to be reduced?

We now examine the expansion of the reduced quadratic irrational, r, into a continued fraction:

$$R = a_1 + \frac{1}{R_2}$$

a_1 is the integer part of r, which means that $\frac{1}{R_2}$ is less than 1.

We want now to examine R_2 to see if it, like R, is a reduced quadratic. The question of whether R is reduced or not involves the conjugate of R_2.

We now prove a lemma (a little proof which is instrumental in proving a more important theorem): Let X be a quadratic irrational, and X' be the conjugate of X, and let h be the integer part of X. Then we have the following:

Lemma. *If X is written $X = h + \frac{1}{Y}$, and if X' is written $X' = h + \frac{1}{Z}$, then, Z is the conjugate of Y.*

Proof. Let

$$X = \frac{a + \sqrt{b}}{c}, \qquad Y = \frac{a - \sqrt{b}}{c}$$

and let h be the integer part of X. We can then write the following:

$$X = \frac{a + \sqrt{b}}{c} = h + \frac{a + \sqrt{b}}{c} - h = h + \frac{a + \sqrt{b} - ch}{c} =$$

$$h + \frac{a - ch + \sqrt{b}}{c} = h + \cfrac{1}{\cfrac{c}{a - ch + \sqrt{b}}} =$$

$$h + \cfrac{1}{\cfrac{c[(a - ch) - \sqrt{b}]}{(a - ch)^2 - b}} = h + \cfrac{1}{\cfrac{(a - ch) - \sqrt{b}}{\cfrac{(a - ch)^2 - b}{c}}}.$$

Therefore, $X = h + \frac{1}{Y}$ and $Y = \cfrac{(a - ch) - \sqrt{b}}{\cfrac{(a - ch)^2 - b}{c}}$.

$$X' = \frac{a - \sqrt{b}}{c} = h + \frac{a - \sqrt{b}}{c} - h = h + \frac{a - \sqrt{b} - ch}{c} =$$

$$h + \cfrac{1}{\cfrac{c}{(a - ch) - \sqrt{b}}} = h + \cfrac{1}{\cfrac{c[(a - ch) + \sqrt{b}]}{(a - ch)^2 - b}} =$$

$$h + \cfrac{1}{\cfrac{(a - ch) + \sqrt{b}}{\cfrac{(a - ch)^2 - b}{c}}};$$

therefore, $X' = h + \frac{1}{Z}$ and $Z = \cfrac{(a - ch) + \sqrt{b}}{\cfrac{(a - ch)^2 - b}{c}}$.

Now by comparison we can see that Z is the conjugate of Y. This completes the proof of the lemma.

We now proceed with our investigation of the expansion of R (a reduced quadratic) into a continued fraction. We can now state

$$R' = a_1 + \frac{1}{R_2}.$$

Now solving for R_2':

$$R'R_2' = a_1R_2' + 1$$
$$R'R_2' - a_1R_2' = 1$$
$$R_2'(R' - a_1) = 1$$
$$R_2' = \frac{1}{R' - a_1}$$
$$R_2' = -\frac{1}{a_1 - R'}.$$

$R' < 0$ (by definition of a reduced quadratic), so R' is a negative number. But a_1 is a positive integer; therefore $a_1 - R' > 1$ and, since $R_2' = -\frac{1}{a_1 - R'}$, we have the result that R_2' is negative. Thus one of the requirements for R_2 being reduced is satisfied, but we must also show that $R_2' > -1$.

Now $R_2' > -1$ because $R_2' = -\frac{1}{a_1 - R'}$ and, as we just showed, $a_1 - R' > 1$. So we can now say that $R_2 > 0$, and $-1 < R_2' < 0$; which means that R_2 is reduced. The same argument could now be applied to R_2 to show that R_3 is also reduced, and to R_3, etc.

We have now shown that each $u_n = \frac{A_n + \sqrt{B}}{C_n}$ which appears in the continued fraction of the reduced quadratic R is itself reduced. Remember that B is the same in each of these expressions, and also that we discovered earlier that for a given B there are only a limited number of possible integral values for A and for C. Therefore if we carry out the expansion of $\frac{A + \sqrt{B}}{C}$ far enough we are bound to come to some pair of values for A and C that has appeared before, and from that point on the terms will repeat. Thus, we have proved the following theorem:

THEOREM 8. *The continued fraction for a reduced quadratic irrational will be a repeating continued fraction.*

We must now prove that this continued fraction is initially repeating. The plan here will be to show that if for two terms (a_n and a_m) it is true that $a_n = a_m$; then, it will be true that $a_{n-1} = a_{m-1}$. If this is true, $a_{n-2} = a_{m-2}$; and finally we will have the result that a_1 is equal to some following term. If this is true, it follows that the continued fraction for a reduced quadratic is initially repeating.

We shall begin the investigation by concentrating our attention on two equal terms, a_n and a_m, which are equal, and by making the following observations:

Since $a_n = a_m$ we can write $u_n = a_n + \frac{1}{u_{n+1}}$. Also according to the lemma we have $u_n' = a_n + \frac{1}{u_{n+1}'}$.

Now let us examine closely the second equation. We want to show that $u_{m-1} = u_{n-1}$. Now $u_{m-1} = a_{m-1} + \frac{1}{u_m}$ and $u_{n-1} = a_{n-1} + \frac{1}{u_n}$. Since $u_m = u_n$, it follows that $\frac{1}{u_m} = \frac{1}{u_n}$. We see that all we have left to do is demonstrate that $a_{m-1} = a_{n-1}$.

Consider the following equations:

$$u_n' = a_n + \frac{1}{u_{n+1}'}$$

$$-\frac{1}{u_{n+1}'} = a_n - u_n'$$

$$-\frac{1}{u_{n+1}'} = a_n + \frac{1}{-\frac{1}{u_n'}}.$$

Since all of the u's are reduced, u_{n+1}' and u_n' both lie between -1 and 0; therefore $-\frac{1}{u_{n+1}'}$ and $-\frac{1}{u_n'}$ are both greater than 0, and $\frac{1}{-\frac{1}{u_n'}}$ is positive but less than 1. It then follows from Equation (1) that a_n is the integer part of $-\frac{1}{u_{n+1}'}$. We assumed at the beginning that $u_m = u_n$. If this is so $u_m' = u_n'$, and $-\frac{1}{u_m'} = -\frac{1}{u_n'}$; and since in general a_n is the integer part of $-\frac{1}{u_{n+1}'}$, we can say that a_{n-1} is the integer part of $-\frac{1}{u_n'}$. Then, also, a_{m-1} is the integer part of $-\frac{1}{u_m'}$. Therefore $a_{n-1} = a_{m-1}$, which was all we needed to show that if $a_n = a_m$ then $a_{m-1} = a_{n-1}$. Thus $a_{m-2} = a_{n-2}$, and finally some a will equal a_1.

We have now proved:

THEOREM 9. *The continued fraction expansion of any reduced quadratic irrational is initially repeating.*

Exercise Set 21

Check the following quadratic irrationals to see which are reduced, and then find the terms of the reduced quadratics by using a term table or by the three-step process to see that the terms are initially repeating.

1. $\frac{3+\sqrt{17}}{2}$ 2. $\frac{1+\sqrt{39}}{2}$ 3. $\frac{-1+\sqrt{17}}{3}$ 4. $\frac{2+\sqrt{12}}{4}$

A SYMMETRIC SEQUENCE

We now turn our attention to a symmetric sequence. A symmetric sequence of terms is *a sequence that is unchanged if the terms of the sequence are written in reverse order.* The sequence

$$a_1, a_2, a_3, \ldots a_{n-2}, a_{n-1}, a_n$$

is symmetric if

$$a_1 = a_n, a_2 = a_{n-1}, a_3 = a_{n-2}, \text{etc.}$$

Here are two examples of symmetric sequences:

EXAMPLE 1. 1, 2, 3, 4, 3, 2, 1.

EXAMPLE 2. 7, 1, 1, 9, 9, 1, 1, 7.

Look once more at your term tables for numbers of the form $\sqrt{b}$. You will notice that in each case the terms form the following pattern:

$$a_1, a_2, a_3, \cdots a_3, a_2, 2a_1, a_2, a_3, \cdots a_3, a_2, 2a_1, a_2,$$

The sequence of terms for a pure quadratic irrational start with a_1; a_1 is followed by a symmetric sequence, which is in turn followed by the term $2a_1$. Examples are now given:

The terms of $\sqrt{19}$ are 4, 2, 1, 3, 1, 2, 8, 2, 1, 3, 1, 2, 8, 2, ...

The terms of $\sqrt{29}$ are 5, 2, 1, 1, 2, 10, 2, 1, 1, 10, 2, ...

Can you explain, at this point, why this should be true? That this will always be true can be established upon the ideas that have been presented earlier in this chapter.

Note, first, that no quadratic irrational number of the form $\sqrt{b}$ is reduced; but, if we add the integer part of $\sqrt{b}$ to $\sqrt{b}$, we have formed a reduced quadratic irrational. This will be clear to you if you observe the following example carefully.

EXAMPLE. $\sqrt{5}$ is not reduced because its conjugate, $-\sqrt{5}$, is not greater than -1. But the integer part of $\sqrt{5}$ is 2, and $2+\sqrt{5}$ is reduced because $2+\sqrt{5}$ is greater than 1, and $2-\sqrt{5}$ is negative but greater than -1.

We have proved that the continued fraction for a reduced quadratic is initially repeating. Also recall that the number represented by the continued fraction formed by reversing the terms of an initially repeating continued fraction, R, is $-\frac{1}{R'}$ where R' is the conjugate of R.

Now let $\sqrt{B}$ be some pure quadratic irrational, then

$$\sqrt{B} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + \cfrac{1}{a_n + \cfrac{1}{2a_1 + \cfrac{1}{a_2 + \cdots}}}}}$$

Note that a_1 is the integer part of $\sqrt{B}$ and, as we stated earlier, $\sqrt{B} + a_1$ is reduced. Then its continued fraction is initially repeating.

$$\sqrt{B} + a_1 = a_1 + a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + \cfrac{1}{a_n + \cfrac{1}{2a_1 + \cfrac{1}{a_2 + \cdots}}}}}$$

Eq. 1. $$\sqrt{B} + a_1 = 2a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + \cfrac{1}{a_n + \cfrac{1}{2a_1 + \cfrac{1}{a_2 + \cdots}}}}}$$

The conjugate of $\sqrt{B} + a_1$ is $a_1 - \sqrt{B}$ and $-\frac{1}{a_1 - \sqrt{B}} = \frac{1}{\sqrt{B} - a_1}$.

Now reverse the repeating terms of $\sqrt{B} + a_1$.

$$\frac{1}{\sqrt{B} - a_1} = a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{a_{n-2} + \cdots + \cfrac{1}{a_2 + \cfrac{1}{2a_1 + \cfrac{1}{a_n + \cdots}}}}}$$

$$\sqrt{B} + a_1 = 2a_1 + \sqrt{B} - a_1$$

Eq. 2. $$\sqrt{B} + a_1 = 2a_1 + \cfrac{1}{\cfrac{1}{\sqrt{B} - a_1}}$$

Observing Equation 1, we see that the continued fraction

$$a_2 + \cfrac{1}{a_3 + \dots + \cfrac{1}{a_n + \cfrac{1}{2a_1 + \cdots}}}$$

plays the same part as $\frac{1}{\sqrt{B} - a_1}$ in Equation 2; therefore,

$$\frac{1}{\sqrt{B} - a_1} = a_2 + \cfrac{1}{a_3 + \dots + \cfrac{1}{a_n + \cfrac{1}{2a_1 + \cdots}}}$$

$$\frac{1}{\sqrt{B} - a_1} = a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{a_{n-2} + \dots + \cfrac{1}{a_2 + \cdots}}} =$$

$$a_2 + \cfrac{1}{a_3 + \dots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n + \cdots}}}$$

Now since these last two continued fractions represent the same number, they must be equal, which in turn means that their corresponding terms are equal. The repeating sequence of terms is the same when reversed so it is a symmetric sequence, and we have

$$a_n = a_2, \qquad a_3 = a_{n-1}, \qquad \text{etc.}$$

Our conclusion is that the repeating sequence of terms of the continued fraction of any number of the form $\sqrt{b}$ is symmetric, except that instead of the first term being twice the integer part of $\sqrt{b}$, it is exactly the integer part of $\sqrt{b}$.

LAGRANGE'S THEOREM

Still another important theorem can be proved about quadratic irrationals:

> LAGRANGE'S THEOREM. *The continued fraction for every quadratic irrational is a repeating continued fraction.*

Lagrange was a French mathematician who lived from 1736 to 1813. He made contributions to many areas of mathematics, particularly to the theory of numbers. He was also known as an astronomer.[9]

No proof of Lagrange's theorem will be given here, but you can find a proof in almost any book dealing with the theory of numbers. We have already proved that the continued fraction for any reduced quadratic will be initially repeating, so all you have do to prove Lagrange's theorem is to show that in the expansion of a number of the form $\frac{A + \sqrt{B}}{C}$ into a continued fraction, one of the expressions of the form $\frac{A_n + \sqrt{B}}{C_n}$ which arises in the three-step process will be reduced. Then from this point on all of these expressions will be reduced quadratics, and the original quadratic irrational will repeat. Why don't you try to prove it?

[9] Fink, Karl. *A Brief History of Mathematics.* London: The Open Court Publishing Co., 1910. p. 312.

CHAPTER 11

OTHER INTERESTING FACTS ABOUT CONTINUED FRACTIONS

AN UNANSWERED QUESTION

As you have seen, mathematicians have studied many properties of quadratic irrational numbers; however, irrationals involving cube roots, fifth roots, sixth roots, etc., are much more difficult to investigate. For instance, it is known that the first few terms of the continued fraction expansion of $\sqrt[3]{2}$ appear as follows:

$$\sqrt[3]{2} = 1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cdots}}}}}}}$$

and it is not known whether or not there is any limit to how large the terms will become.[10]

[10] Davenport, H. *The Higher Arithmetic.* London: Hutchinson's University Library, 1952. p. 107.

DISCOVERING A FIBONACCI SEQUENCE

Doesn't it seem to you that there ought to be something special about the following continued fraction?

$$0 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1} + \cdots}}}}}$$

Let us see if we can discover anything interesting about it. We might as well start by evaluating it. Setting the continued fraction equal to y, we get

$$y = 0 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + y}}} \qquad \text{or} \qquad y = 0 + \frac{1}{1 + y}.$$

Solving the second expression for y, we have

$$y + y^2 = 1$$
$$y^2 + y - 1 = 0$$

Now applying the quadratic formula for y:

$$y = \frac{-1 \pm \sqrt{1 - 4(1)(-1)}}{2\cdot 1}$$

$$y = \frac{-1 \pm \sqrt{5}}{2}.$$

But y is positive, so

$$y = \frac{-1 + \sqrt{5}}{2}.$$

Using $\sqrt{5} = 2.236+$ and evaluating y, gives:

$$y = \frac{-1 + 2.236 +}{2}.$$

$$y = 0.618+.$$

Now let us place the terms of the continued fraction in a convergent table and see what happens.

TABLE XXVI

n	-1	0	1	2	3	4	5	6	7	8	9	10
a_n			0	1	1	1	1	1	1	1	1	1
r_n	0	1	0	1	1	2	3	5	8	13	21	34
s_n	1	0	1	1	2	3	5	8	13	21	34	55

Note that the same sequence of numbers appears in both the r-row and the s-row of the table. Each row contains the sequence 1, 1, 2, 3, 5, 8, 13, 21, ... in which each term is obtained by adding together the two previous terms. This sequence of numbers is known as the *Fibonacci sequence* and the terms in the sequence are called *Fibonacci numbers*. Fibonacci was an Italian mathematician of the 13th century.

The Fibonacci sequence occurs repeatedly in nature. For example, buds form a spiral as they appear on a twig of a tree, or on a bush, or weed; and the number of buds in a spiral is always one of the numbers in the Fibonacci sequence.

THE GOLDEN RATIO

Evaluating C_{10} we get the following:

$$C_{10} = \frac{r_{10}}{s_{10}} = \frac{34}{55} = 0.618+.$$

The number 0.618+, which is approximately the value of $\dfrac{-1+\sqrt{5}}{2}$ to which each convergent is getting closer, is known as the *golden section* or the *golden ratio*; and a rectangle in which the ratio of the width to the length is near the golden ratio is said to be the "most beautiful rectangle." It is easy to see that this number has had some influence in the development of art.

CONTINUED FRACTIONS AND GEOMETRY

We will now consider a relationship between continued fractions and geometry. We shall use continued fractions to prove that $\sqrt{2}$ is irrational. Consider Figure 1.

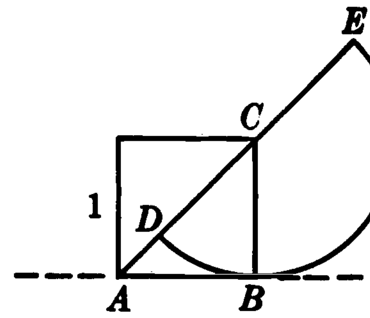

FIGURE 1

Given: A square whose side is unity and with arcs drawn as indicated. $AC^2 = AB^2 + BC^2 = 1^2 + 1^2 = 2$. Therefore $AC = \sqrt{2}$.

$$\frac{AC}{BC} = \frac{\sqrt{2}}{1}$$

$$\sqrt{2} = \frac{AC}{BC} = \frac{CD + AD}{BC}$$

$$= 1 + \frac{AD}{BC}$$

$$= 1 + \cfrac{1}{\cfrac{BC}{AD}}$$

$$= 1 + \cfrac{1}{2 + \cfrac{AD}{AB}} \quad \text{(See } \textit{note.}\text{)}$$

$$= 1 + \cfrac{1}{2 + \cfrac{1}{\cfrac{AB}{AD}}}$$

$$= 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{AD}{AB}}}$$

We can see now that this will be a non-terminating continued fraction and, as we noted earlier, a non-terminating continued fraction represents an irrational number.

Note: AD and AE are segments of a secant through the circle with center at C. AB is tangent to the arc with center at C. Therefore from plane geometry we have the following:

$AB^2 = AE \cdot AD$, or $\frac{AB}{AD} = \frac{AE}{AB}$;

$AE = AD + DE = AD + 2BC$,

and $BC = AB$;

$$\frac{BC}{AD} = \frac{AB}{AD} = \frac{AE}{AB} = \frac{AD + 2BC}{AB} = \frac{AD + 2AB}{AB} = 2 + \frac{AD}{AB}.$$

APPENDIX A

PROOFS OF SELECTED THEOREMS

THEOREMS REFERRED TO BUT NOT PROVED IN PREVIOUS CHAPTERS

Proof No. 1

Our objective is to prove the following:

THEOREM. *Every rational number can be expanded into a terminating continued fraction.*

Consider a rational number $\frac{p}{q}$. Dividing: $\frac{p}{q} = a_1 + \frac{r_1}{q_1}$, where a_1 is the largest integer less than or equal to $\frac{p}{q}$. If a_1 is equal to $\frac{p}{q}$ the division is finished, and the continued fraction is certainly terminating. If a_1 is not equal to $\frac{p}{q}$, then r_1 is a positive number but less than q. Similarly: $\frac{q}{r_1} = a_2 + \frac{r_2}{r_1}$, and $\frac{r_1}{r_2} = a_3 + \frac{r_3}{r_2}$. In each case the a_n is the greatest integer in the corresponding fraction, and r_{n+1} is less than r_n. The r's are positive integers that decrease with each step. Therefore an r (a remainder) of zero will appear since there are only a limited number of positive integers less than a given integer. When a remainder of zero is obtained, the continued fraction expansion stops.

Proof No. 2

This will be a proof by mathematical induction to show that the formulas

$$r_n = a_n r_{n-1} + r_{n-2} \quad \text{and} \quad s_n = a_n s_{n-1} + s_{n-2}$$

are true for all values of n. Here r_n and s_n are respectively the numerator and denominator of the nth convergent of the continued fraction of the rational number $\frac{r}{s}$, and a_n is the nth term. We showed in Chapter 2 that these formulas are valid for $n = 1$, $n = 2$, $n = 3$, and $n = 4$. We must now show that every time the formulas are true for a particular value of n they are true for the next value of n.

We assume that the formulas are true for $n = m$, then we have

$$r_m = a_m r_{m-1} + r_{m-2} \quad \text{and} \quad s_m = a_m s_{m-1} + s_{m-2}.$$

This assumption is made in accordance with what is called the *induction hypothesis.*

We must now show that

$$r_{m+1} = a_{m+1} r_m + r_{m-1} \quad \text{and} \quad s_{m+1} = a_{m+1} s_m + s_{m-1}.$$

We want to know the value of the following:

$$C_{m+1} = \frac{r_{m+1}}{s_{m+1}} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + \cfrac{1}{a_{m-1} + \cfrac{1}{a_m + \cfrac{1}{a_{m+1}}}}}}$$

Here we have $m + 1$ terms, but if we consider $a_m + \frac{1}{a_{m+1}}$ as being only one term we have the following:

$$C_{m+1} = \frac{r_{m+1}}{s_{m+1}} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + \cfrac{1}{a_{m-1} + \cfrac{1}{\left(a_m + \cfrac{1}{a_{m+1}}\right)}}}}$$

Using the induction hypothesis we can write

$$C_m = \frac{r_m}{s_m} = \frac{a_m r_{m-1} + r_{m-2}}{a_m s_{m-1} + s_{m-2}}.$$

We assumed this formula to be valid for a continued fraction having m terms. In order to evaluate C_{m+1} using this formula, we can consider $a_m + \dfrac{1}{a_{m+1}}$ as being only one (the mth) term. According to this observation C_{m+1} has only m terms, and we can evaluate it by substituting $a_m + \dfrac{1}{a_{m+1}}$ for a_m as follows:

$$C_{m+1} = \frac{r_{m+1}}{s_{m+1}} = \frac{\left(a_m + \dfrac{1}{a_{m+1}}\right) r_{m-1} + r_{m-2}}{\left(a_m + \dfrac{1}{a_{m+1}}\right) s_{m-1} + s_{m-2}}$$

$$= \frac{\dfrac{(a_m a_{m+1} + 1)\, r_{m-1}}{a_{m+1}} + r_{m-2}}{\dfrac{(a_m a_{m+1} + 1)\, s_{m-1}}{a_{m+1}} + s_{m-2}}$$

Multiply both numerator and denominator of this expression by a_{m+1}.

$$= \frac{a_m a_{m+1} + 1\, r_{m-1} + a_{m+1}\, r_{m-2}}{a_m a_{m+1} + 1\, s_{m-1} + a_{m+1}\, s_{m-2}}$$

$$= \frac{a_m a_{m+1} r_{m-1} + r_{m-1} + a_{m+1} r_{m-2}}{a_m a_{m+1} s_{m-1} + s_{m-1} + a_{m+1} s_{m-2}}$$

$$= \frac{a_{m+1}(a_m r_{m-1} + r_{m-2}) + r_{m-1}}{a_{m+1}(a_m s_{m-1} + s_{m-2}) + s_{m-1}}.$$

Note: From the induction hypothesis we have

$$a_m r_{m-1} + r_{m-2} = r_m$$

and

$$a_m s_{m-1} + s_{m-2} = s_m.$$

Making these substitutions in the last expression for C_{m+1}, above, we have

$$C_{m+1} = \frac{r_{m+1}}{s_{m+1}} = \frac{a_{m+1} r_m + r_{m-1}}{a_{m+1} s_m + s_{m-1}},$$

and by definition r_{m+1} and s_{m+1} are respectively the numerator and denominator of the $m+1$th convergent, C_{m+1}. Therefore,

$$r_{m+1} = a_{m+1} r_m + r_{m-1} \qquad \text{and} \qquad s_{m+1} = a_{m+1} s_m + s_{m-1}.$$

This completes the proof.

Proof No. 3

This will also be a proof by mathematical induction. The object will be to prove that the equation

$$r_n s_{n-1} - r_{n-1} s_n = (-1)^n$$

is true for all values of n.

We have shown in Chapter 2 that the equation is true for $n = 1, n = 2,$ and $n = 3$. Let us assume that the formula is true for some integer m, with m being greater than 3. Then our induction hypothesis is

$$r_m s_{m-1} - r_{m-1}s_m = (-1)^m.$$

Now we need to show that $\quad r_{m+1}s_{(m-1+1)} - r_{(m-1+1)}\,s_{m+1} = (-1)^{m+1}$
or $\quad r_{m+1}s_m - r_m s_{m+1} = (-1)^{m+1}.$

Let us work with this relationship (which is not proved) to see if we can discover a relationship which is simpler and might give us a clue as to how to proceed with the proof.

$$r_{m+1}s_m - r_m s_{m+1} \stackrel{?}{=} (-1)^{m+1}$$

$$r_{m+1}s_m - r_m s_{m+1} \stackrel{?}{=} (-1)(-1)^m$$

$$-r_{m+1}s_m + r_m s_{m+1} \stackrel{?}{=} (-1)^m$$

$$r_m s_{m+1} - r_{m+1}s_m \stackrel{?}{=} (-1)^m$$

From the induction hypothesis, $r_m s_{m-1} - r_{m-1}s_m = (-1)^m$; so make the equivalent value a substitution for $(-1)^m$:

$$r_m s_{m+1} - r_{m+1}s_m \stackrel{?}{=} r_m s_{m-1} - r_{m-1}s_m$$

$$r_m(s_{m-1} - s_{m+1}) \stackrel{?}{=} s_m(r_{m-1} + r_{m+1})$$

$$\frac{r_m}{s_m} \stackrel{?}{=} \frac{r_{m-1} + r_{m+1}}{s_{m-1} + s_{m+1}}$$

$$\frac{r_m}{s_m} \stackrel{?}{=} \frac{r_{m-1} - (a_{m+1}r_m + r_{m-1})}{s_{m-1} - (a_{m+1}s_m + s_{m-1})}$$

$$\frac{r_m}{s_m} \stackrel{?}{=} \frac{r_{m-1} - a_{m+1}r_m - r_{m-1}}{s_{m-1} - a_{m+1}s_m - s_{m-1}}$$

$$\frac{r_m}{s_m} \stackrel{?}{=} \frac{-a_{m+1}r_m}{-a_{m+1}s_m}$$

$$\frac{r_m}{s_m} = \frac{r_m}{r_m}.\quad \text{Finally we have a true statement.}$$

All we have now found is that: If $r_{m+1}s_m - r_m s_{n-1} = (-1)^{n+1}$ is true, then $\frac{r_m}{s_m} = \frac{r_m}{s_m}$. At this point we have proved nothing, because we do not yet know if $r_{m+1}s_m - r_m s_{m+1} = (-1)^{m+1}$. However, $\frac{r_m}{s_m} = \frac{r_m}{s_m}$ is true. So if we can prove the statement, "if $\frac{r_m}{s_m} = \frac{r_m}{s_m}$ then $r_{m+1}s_m - r_m s_{n+1} = (-1)^{m+1}$", we will have proved that $r_{m+1}s_m - r_m s_{m+1} =$

$(-1)^{m+1}$ because it is true that $\frac{r_m}{s_m} = \frac{r_m}{s_m}$. Then *if* we can retrace our steps, we will get the desired results:

$$\frac{r_m}{s_m} = \frac{r_m}{s_m},$$

$$\frac{r_m}{s^m} = \frac{-a_{m+1}r_m}{-a_{m+1}s_m},$$

$$\frac{r_m}{s_m} = \frac{r_{m-1} - a_{m+1}r_m - r_{m-1}}{s_{m-1} - a_{m+1}s_m - s_{m-1}},$$

$$\frac{r_m}{s_m} = \frac{r_{m-1} - (a_{m+1}r_m + r_{m-1})}{s_{m-1} - (a_{m+1}s_m + s_{m-1})},$$

$$\frac{r_m}{s_m} = \frac{r_{m-1} - r_{m+1}}{s_{m-1} - s_{m+1}},$$

$$r_m(s_{m-1} - s_{m+1}) = s_m(r_{m-1} - r_{m+1}),$$

$$r_m s_{m-1} - r_m s_{m+1} = s_m r_{m-1} - s_m r_{m+1},$$

$$r_m s_{m-1} - r_{m-1}s_m = r_m s_{m+1} - r_{m+1}s_m.$$

We now know, by the induction hypothesis, that $r_m s_{m-1} - r_{m-1}s_m = (-1)^m$. It follows next that, after substituting for $r_m s_{m-1} - r_{m-1}s_m$, we have

$$r_m s_{m+1} - r_{m+1}s_m = (-1)^m,$$

$$(-1)(r_m s_{m+1} - r_{m+1}s_m) = (-1)^m,$$

$$r_{m+1}s_m - r_m s_{m+1} = (-1)^{m+1}.$$

We now know that whenever the formula

$$r_n s_{n-1} - r_{n-1}s_n = (-1)^n$$

is valid for $n = m$, it is also valid for $n = m + 1$. Therefore, since we proved in Chapter 2 that this formula is true for $n = 3$, we know it is true for $n = 4$; and if it is true for $n = 4$, it is true for $n = 5$, etc. It is, then, true for all values of n. Of course, since n is just the number of a term, n will always be a positive integer. Thus the proof is complete.

Proof No. 4

The purpose here is to show that if the equation

$$x^2 - Py^2 = 1$$

is satisfied by the values x_1 and y_1 where x_1 and y_1 are integers, then, $x_1^2 + y_1^2P$ and $2x_1y_1$ are also solutions to the equation.

Let $\qquad x_2 = x_1^2 + y_1^2P \qquad$ and $\qquad y_2 = 2x_1y_1.$

If x_2 and y_2 are really solutions, we should find that $x_2^2 + Py_2^2 = 1$.

$$
\begin{aligned}
x_2^2 - Py_2^2 &= (x_1^2 + y_1^2P)^2 - P(2x_1y_1)^2 \\
&= x_1^4 + 2x_1^2y_1^2 + y_1^4P^2 - 4Px_1^2y_1^2 \\
&= x_1^4 - 2x_1^2y_1^2 + y_1^4P^2 \\
x_2^2 - Py_2^2 &= (x_1^2 - y_1^2P)^2
\end{aligned}
$$

Now since $x_1^2 - y_1^2P = 1$, it follows that $(x_1^2 - y_1^2P)^2 = 1$; therefore, we have $x_2^2 - Py_2^2 = 1$. So $x_2 = x_1^2 + y_1^2P$ and $y_2 = 2x_1y_1$ are also solutions to the equation $x^2 + Py^2 = 1$.

Proof No. 5

This "proof" will consist of one example showing how the repeating decimal 0.1 2 3 7 3 7 $\cdots$ can be written as a rational number.

$$
\begin{aligned}
N &= 0.1\,2\,3\,7\,3\,7\,3\,7 \cdots \\
10{,}000N &= 1\,2\,3\,7.3\,7\,3\,7 \cdots \\
100N &= 1\,2.3\,7\,3\,7 \cdots \\
\text{(subtracting)} \quad 9{,}900N &= 1\,2\,2\,5.0\,0\,0\,0 \\
N &= \frac{1\,2\,2\,5}{9\,9\,0\,0} \quad \text{(a rational number).}
\end{aligned}
$$

APPENDIX B

ANSWERS TO EXERCISES

Set 1:

1. $2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{2}}}}}$

2. $3+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{2}}}}$

3. $2+\cfrac{1}{3+\cfrac{1}{1+\frac{1}{2}}}$

4. $0+\cfrac{1}{1+\cfrac{1}{2+\frac{1}{6}}}$

5. $1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{2}}}}}}}$

6. $0+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{2}}}}}}}}$

7. $3+\cfrac{1}{4+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{5+\cfrac{1}{1+\frac{1}{4}}}}}}$

Set 2:

1. $\frac{41}{24}$ 2. $\frac{9}{56}$ 3. $\frac{225}{157}$

Set 3:

1. $-3 + \cfrac{1}{1 + \cfrac{1}{6}}$ 2. $-2 + \cfrac{1}{4 + \cfrac{1}{3}}$ 3. $-5 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{2}}}}$

Set 4:

1. $2, 3, \frac{8}{3}, \frac{11}{4}, \frac{30}{11}$ 2. $5, \frac{21}{4}, \frac{68}{13}, \frac{157}{30}$ 3. $3, \frac{7}{2}, \frac{31}{9}, \frac{193}{56}, \frac{1382}{401}$

Set 5:

1.

n	−1	0	1	2	3	4
a_n			2	3	1	2
r_n	0	1	2	7	9	25
s_n	1	0	1	3	4	11

2.

n	−1	0	1	2	3	4
a_n			1	2	2	2
r_n	0	1	1	3	7	17
s_n	1	0	1	2	5	12

3.

n	−1	0	1	2	3	4
a_n			3	1	2	3
r_n	0	1	3	4	11	37
s_n	1	0	1	1	3	10

4.

n	−1	0	1	2	3	4
a_n			2	3	1	4
r_n	0	1	2	7	9	43
s_n	1	0	1	3	4	19

5.

n	−1	0	1	2	3	4	5
a_n			0	2	3	1	4
r_n	0	1	0	1	3	4	19
s_n	1	0	1	2	7	9	43

6.

n	−1	0	1	2	3	4
a_n			6	3	2	3
r_n	0	1	6	19	44	151
s_n	1	0	1	3	7	24

7.

n	−1	0	1	2	3	4
a_n			0	5	3	2
r_n	0	1	0	1	3	7
s_n	1	0	1	5	16	37

8.

n	−1	0	1	2	3	4
a_n			1	2	2	2
r_n	0	1	1	3	7	17
s_n	1	0	1	2	5	12

Note: Your work for Exercises 7 and 8, page 10, has shown you that the fractions $\frac{133}{703}$ and $\frac{119}{84}$ can be reduced.

Set 6:

n	−1	0	1	2	3	4	5	6
a_n			2	3	1	3	4	1
r_n	0	1	2	7	9	34	145	179
s_n	1	0	1	3	4	15	64	79

Note: Your answers up to $n = 6$ should read: $-1, +1, -1, +1, -1, +1$.

Set 7:

1. $\frac{1061}{242} - \frac{69}{16} < \left| 0.00055 \right|$ 2. $\frac{984}{181} - \frac{299}{55} < \left| 0.00011 \right|$

Set 8:

1. $x = 10, y = -28.$ 2. $x = 50, y = -12.$ 3. $x = -1, y = -3.$
4. No integer solutions. Note that 217 and 105 have a common divisor which is not a divisor of 6.
5. $m = -400, n = 700.$ 6. $m = 106, n = 31.$

Set 9:

For t, any integer, we have the following:

1. $x = 10 + 11t, y = -28 - 31t.$ If $t = 2$, $x = 32$ and $y = -90.$
2. $x = 50 + 54t, y = -12 - 13t.$ If $t = 3$, $x = 212$ and $y = -51.$
3. $x -1 + 6t, y = -3 + 17t.$ If $t = 1, x = 5$ and $y = 14.$
5. $m = -400 + 19t, n = 700 - 33t.$ If $t = -2, m = -438$ and $n = 766.$
6. $m = 106 + 253t, n = 31 + 74t.$ If $t = -1, m = -147$ and $n = -43.$

Set 10:

For k, any integer, we have the following:

1. 27, or $27 + 5k$. 2. 95, or $95 + 12k$. 3. -42, or $-42 + 9k$.
4. -144, or $-144 + 11k$.

Set 11:

1. $\frac{9}{4}$ 2. $\frac{35}{11}$ 3. $\frac{13}{5}$

Set 12:

1. $\frac{23}{36}$ 2. $\frac{35}{64}$ 3. $\frac{67}{94}$ 4. $\frac{37}{96}$

Set 13:

1. 9 2. 7 3. 3 4. -6 5. 2

Set 14:

The terms of the continued fractions should appear as follows:

1. 3, 3, 6, 3, 6, 3, 6, · · ·
2. 7, 2, 14, 2, 14, · · ·
3. 6, 4, 12, 4, 12, · · ·
4. 8, 1, 7, 1, 16, 1, 7, 1, 16, · · ·

Set 15:

1. 2, 6, 3, 6, 3, 6, 3, · · ·
2. 3, 2, 3, 2, 3, 2, · · ·
3. 3, 1, 2, 2, 2, · · ·

Set 16:

1. $\dfrac{9+\sqrt{21}}{6}$ 2. $\dfrac{5+\sqrt{3}}{2}$ 3. $\dfrac{9+\sqrt{15}}{3}$

Set 17:

1.

n	1	2	3	4	5	6
A_n	0	6	5	5	6	6
C_n	1	11	2	11	1	11
a_n	6	1	5	1	12	1

· · ·

2.

n	1	2	3	4	5	6	7	8
A_n	0	4	2	4	4	2	4	4
C_n	1	6	3	2	3	6	1	6
a_n	4	1	2	4	2	1	8	1

· · ·

3.

n	1	2	3	4	5	6	7	8
A_n	0	8	7	7	3	3	7	7
C_n	1	5	2	10	6	10	2	10
a_n	8	3	7	1	1	1	7	1

· · ·

4.

n	1	2	3	4	5	6	7	8
A_n	3	5	2	3	3	2	5	5
C_n	2	7	5	6	5	7	2	7
a_n	4	1	1	1	1	1	5	1

· · ·

5.

n	1	2	3	4	5	6
A_n	2	1	3	3	1	2
C_n	3	4	1	4	3	3
a_n	1	1	6	1	1	1

· · ·

6.

n	1	2	3	4	5	6	7	8
A_n	−3	3	3	1	4	4	1	3
C_n	4	3	4	5	1	5	4	3
a_n	0	2	1	1	8	1	1	2

· · ·

7.

n	1	2	3	4	5	6	
A_n	−3	7	0	5	5	5	$\cdots$
C_n	−2	7	5	2	5	2	
a_n	−2	1	1	5	2	5	

8.

n	1	2	3	4	5	
A_n	8	8	8	8	8	$\cdots$
C_n	16	2	4	8	4	
a_n	1	8	4	2	4	

Set 18:

1. $x = 8, y = 3$. 2. $x = 55, y = 12$. 3. $x = 35, y = 6$.

4. $x = 649, y = 180$. 5. $x = 9{,}801$; $y = 1{,}820$.

Set 19:

1. $x_2 = 127, y_2 = 48$. 2. $x_2 = 6{,}049$; $y_2 = 1{,}320$.

3. $x_2 = 2{,}449$; $y_2 = 420$.

Set 19a:

1. $x_2 = 49, y_2 = 20$. 2. $x_2 = 97, y_2 = 28$.

3. $x_2 = 51, y_2 = 10$. 4. $x_2 = 2{,}737$; $y_2 = 444$.

Set 20:

1. $\dfrac{1 + \sqrt{3}}{1}$ 2. $\dfrac{29 + \sqrt{1093}}{18}$

Set 21:

1. Reduced; the terms are 3, 1, 1, 3, 1, 1, $\cdots$.
2. Not reduced.
3. Not reduced.
4. Reduced; the terms are 1, 2, 1, 2, 1, 2, $\cdots$.

APPENDIX C

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